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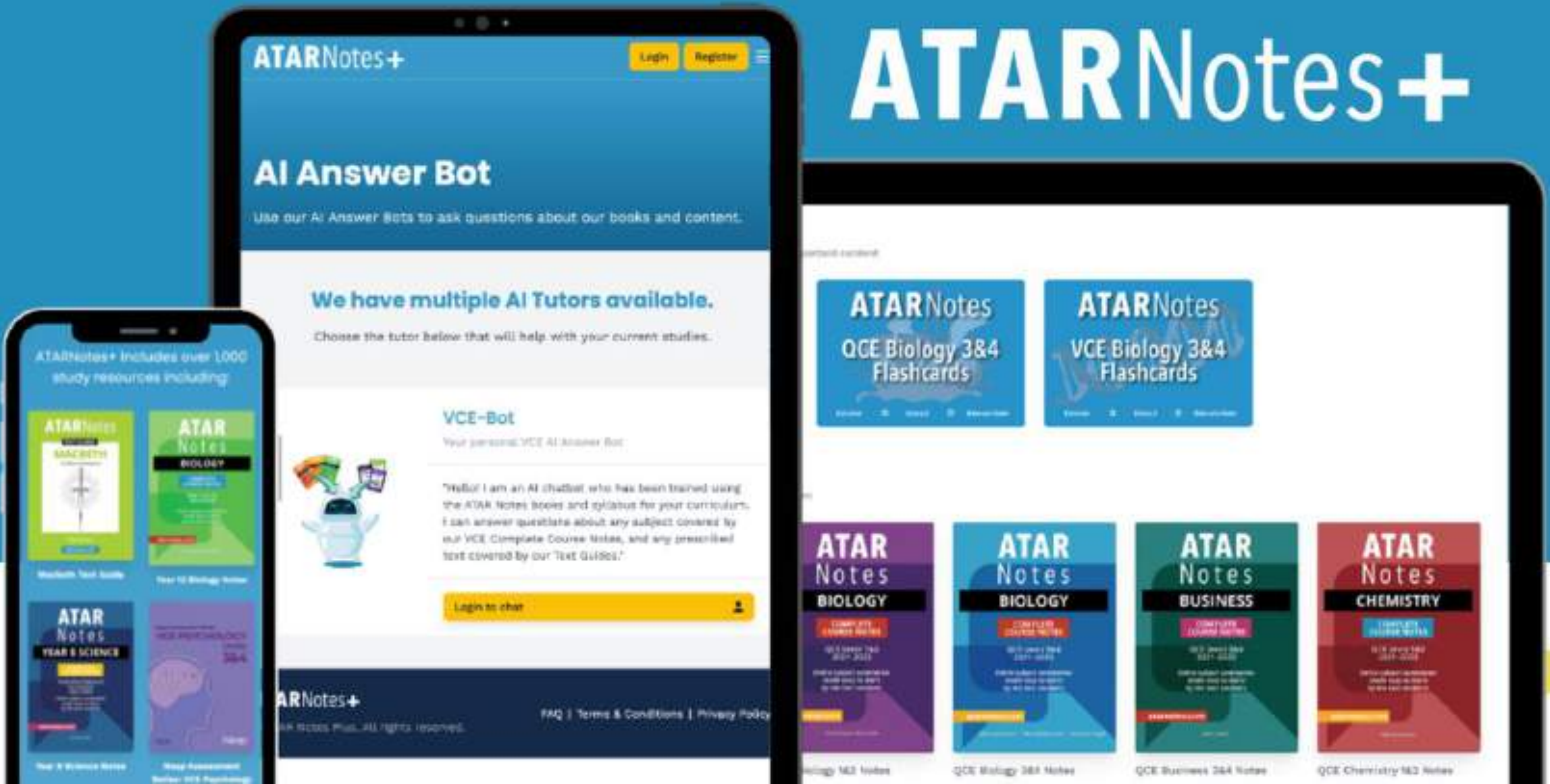


ATARNotes

Maths Methods 3/4

ATARNotes January Lecture Series

Presented by:
Kaif Qais

If you want it straight:

My super simple advice:

- Have a good foundation
  - Use your teachers, peers, textbooks, internet to understand the basic concepts
  - Don’t be afraid to be a nerd! Youtube videos, Wikipedia pages, messing around in Desmos etc.
- **Practice is the only thing that matters** – exposure to questions/time organisation
  - Chapter reviews, past SACs, exam papers, Checkpoints
  - Force your teacher to be on their toes about giving you these
- EXAM PAPERS!!!
  - At least 10 whole papers (exams 1 and 2) – put questions you got wrong + solutions in your bound ref
- Calculator skills save time

Today’s Plan

Topics to be covered

Today will be split up into two blocks:

- Block 1: (Quick ½ Review/Polynomials/Functions)
- Block 2: (Transformations/Logs/Exponentials)
- + Some bonus content at the end!

Announcements

Ask questions in the chat!

Very brief review of linear equations

Some formulas you need to understand from last year:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$y = mx + c \qquad or \qquad y - y_1 = m(x - x_1)$$

Polynomials

When you have an expression with a variable (x) raised to integer powers, such as

- $32x^7 + x^5 - x$
- $\frac{1}{2}x^2 + 2020x - 3$
- $\sqrt{3}$

This topic requires you to: sketch, solve, factorise, find rules

- SUPER IMPORTANT TO MEMORISE SHAPES OF THESE GRAPHS (for degrees 1 to 4)
- The **degree** of a polynomial is the **value of the highest power present**.

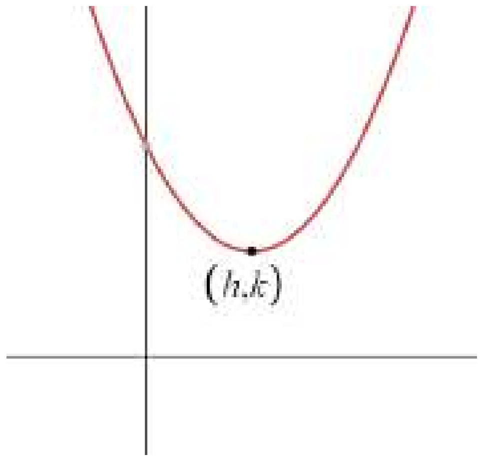
Many forms and methods to determine/solve rules for parabolas

	Turning Point Form	Factorized Form	Expanded Form
Form	$y = a(x-h)^2+k$	$y = a(x-b)(x-c)$	$y = ax^2 + bx + c$
Method to solve for x (when y = 0)	Rearrange for x	Null factor theorem	Quadratic Formula (or convert to other forms)
Key info given in question / form	Turning point (h, k)	x-intercepts b and c	y-intercept at c

Turning point form:

$$y = a(x-h)^2 + k$$

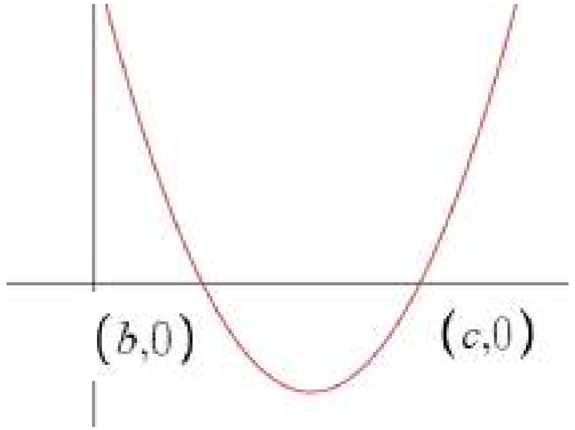
- Turning point shown by (h, k)
- If a > 0 minimum (think ☺)
- If a < 0 maximum (think ☹)



Factorised form:

$$y = a(x-b)(x-c)$$

- Shows where it touches the x axis: $(b, 0)$ $(c, 0)$



	Inflection Point Form	Factorized Form	Expanded Form
Form	$y = a(x-h)^3 + k$	$y = a(x-b)(x-c)(\mathrm{x}-\mathrm{d})$	$y = ax^3 + bx^2 + cx + d$
Method to solve for x (when y = 0)	Rearrange for x	Null factor theorem	Factorise via long division (use rational root theorem then factor theorem)
Key info given in question / form	Inflection point (h, k)	x-intercepts b, c and d	y-intercept at d

Polynomials

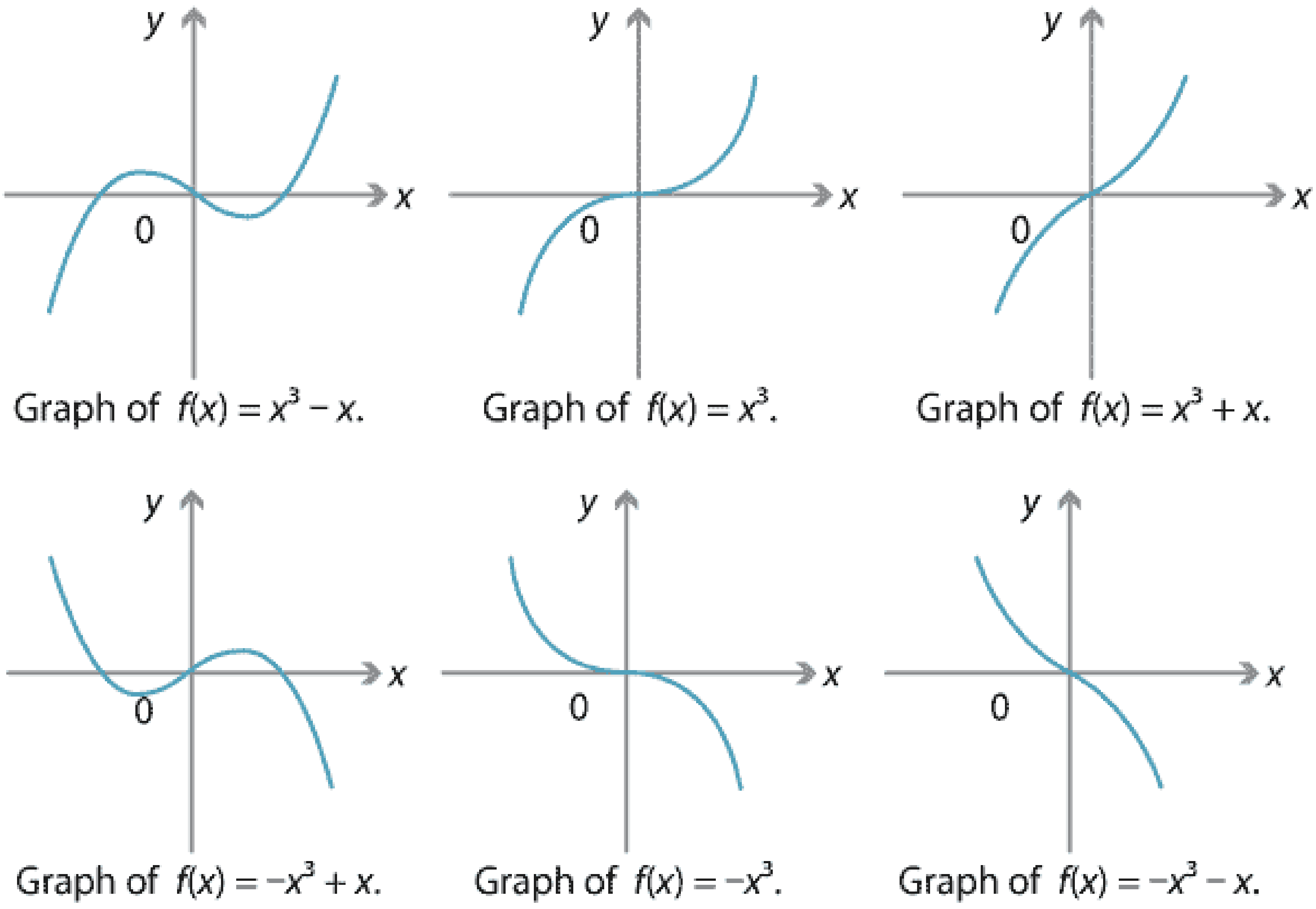
Cubics

Same concepts but:

- Turning pts but also:
  - Stationary pt of inflection (SPOI)
  - Pt of inflection (POI)
    - Difference: SPOI actually has a gradient of 0. POIs are just changes in the slope of gradient
  - Found using calculus

Positive cubic is

- MOUNTAIN then VALLEY



Graph of $f(x) = x^3 - x$.

Graph of $f(x) = x^3$.

Graph of $f(x) = x^3 + x$.

Graph of $f(x) = -x^3 + x$.

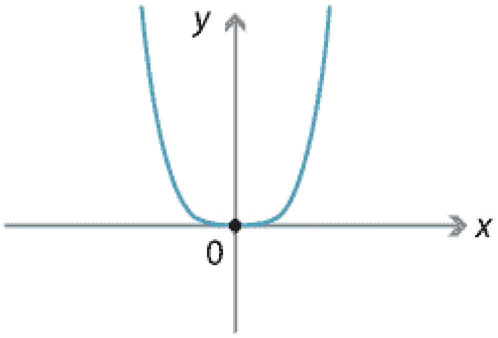
Graph of $f(x) = -x^3$.

Graph of $f(x) = -x^3 - x$.

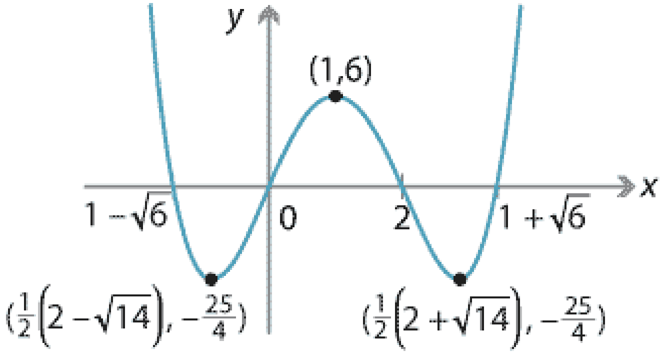
Quartics

It's a quadratic but with a <u>squarer</u> base

- Can have turning points and SPOIs/POIs



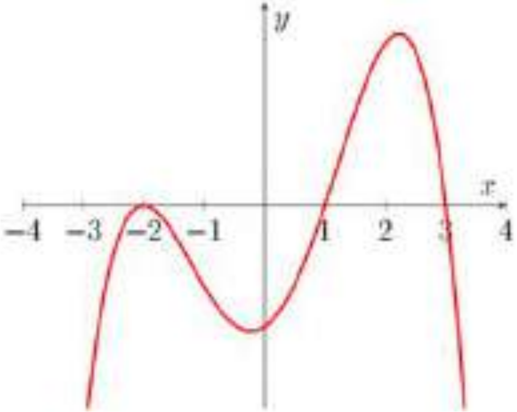
Graph of $f(x) = x^4$.

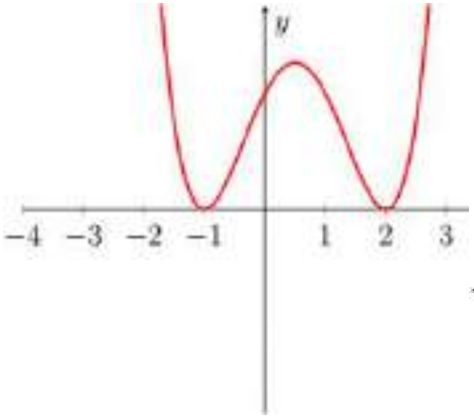


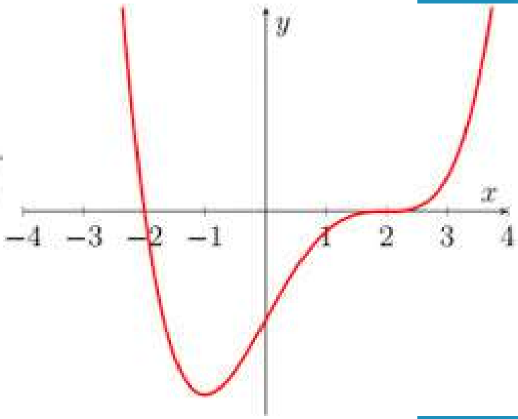
Graph of $f(x) = x^4 - 4x^3 - x^2 + 10x$.

Positive quartic:

- U or W (we get these ws)



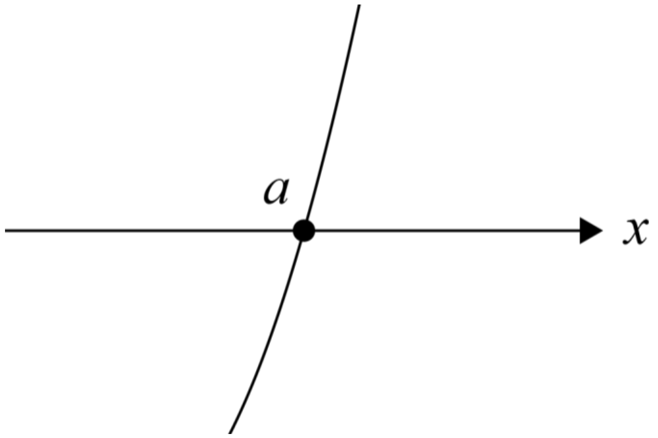




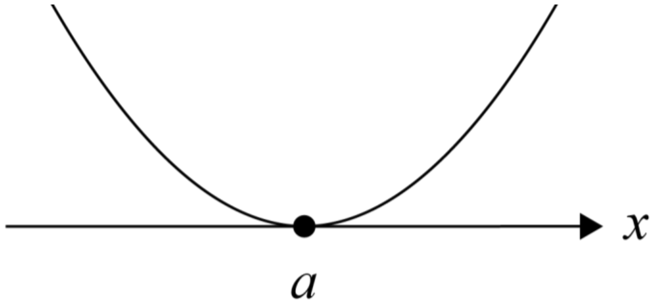
Sketching polynomials

Depending on the power of the factor = look like the shape

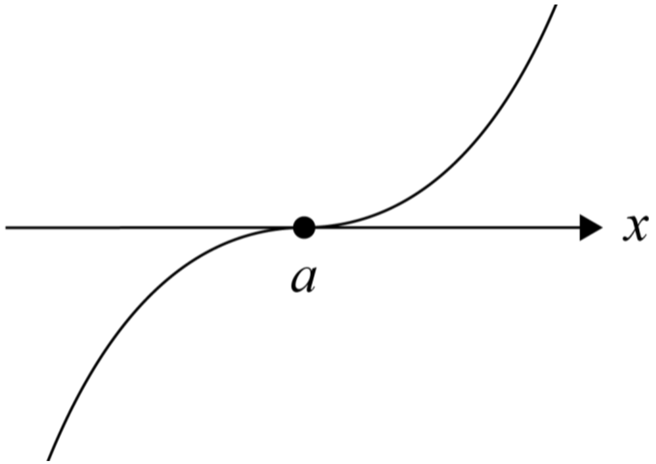
Factor: $(x - a)$



Factor: $(x - a)^2$



Factor: $(x - a)^3$



These are the only kind of inflection points you will have to worry about

Polynomials

Sketching polynomials

Sketching checklist: $y = x^3 - 4x$

Face value info

y-intercept $= 0$

x-intercepts:

x-intercept: when $y = 0$,

$$0 = x^3 - x$$
$$= x(x^2 - 1)$$
$$0 = x(x+1)(x-1)$$
$$x = -1 \text{ or } x = 0 \text{ or } x = 1$$

Turning points:

stationary points:

$$\frac{dy}{dx} = 3x^2 - 1$$

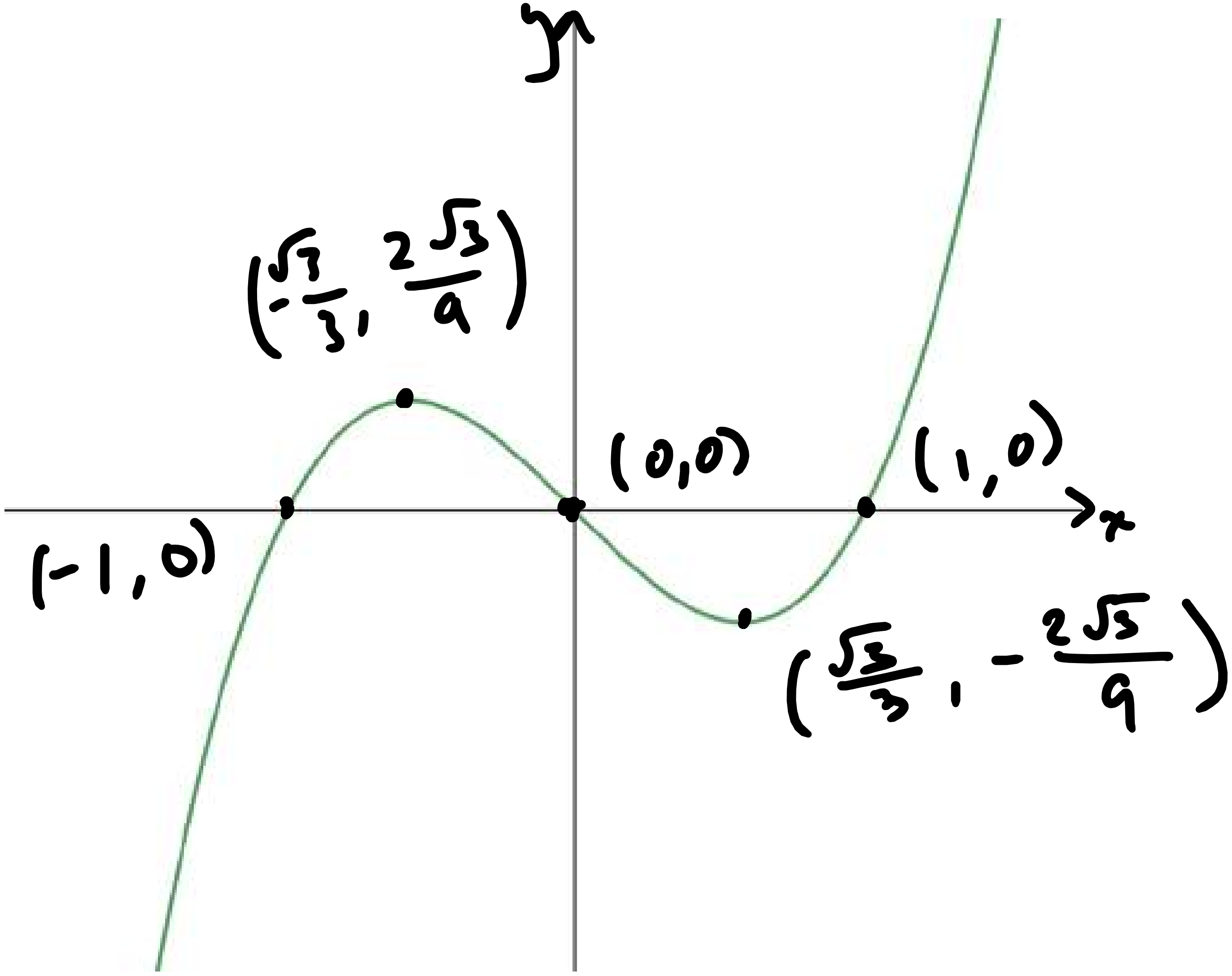
at stationary points, $\frac{dy}{dx} = 0$:

$$0 = 3x^2 - 1$$
$$x = \pm \frac{\sqrt{3}}{3}$$

when $x = \frac{\sqrt{3}}{3}$, $y = \frac{2\sqrt{3}}{9}$

when $x = -\frac{\sqrt{3}}{3}$, $y = -\frac{2\sqrt{3}}{9}$

$\therefore$ stationary points at $\left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$...



y
x
$\left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$
(0,0)
(1,0)
(-1,0)
$\left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$

MEMORISE THESE!

For Cubics:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

For ANY perfect polynomial:

Use pascals triangle!

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Factorising polynomials

To factorise a polynomial in expanded form, we apply the **rational root theorem**, then the **factor theorem**.

- Rational root theorem: for a polynomial equation $0 = ax^3 + bx^2 + cx + d$, possible solutions are given by $x = \frac{d'}{a'}$, where d' is all the factors of d and a is all the factors of a
  - For instance, possible solutions to $0 = 3x^3 - 1$ are ± 1 or $\pm \frac{1}{3}$
- Factor theorem: for a given polynomial p, if $p\left(-\frac{a}{b}\right) = 0$, then $ax + b$ is a factor
  - For instance, if $p\left(\frac{1}{2}\right) = 0$ for a given polynomial, then $2x - 1$ is a factor

If **two polynomials are equal**, then **each matching coefficient must be the same**.

For example:

$$\underline{x^2} - \underline{4} = \underline{a}x^2 + \underline{b}x + \underline{c}$$

This means that:

- coefficients of x^2 terms are equal (a=1)
- coefficients of x terms are equal (b=0)
- coefficients of x^0 terms are equal (c=-4)

Factorising polynomials

We can use this to factorise polynomials!

We could also use long division… but that's too LONG (and full of mistakes)

1. Substitute guesses for x until $\mathrm{p}(x) = 0$
   - Use the rational root theorem to narrow down your guesses
2. Rewrite the polynomial as the product of two factors using the factor theorem
3. Multiply and equate coefficients
4. Factorise the final quadratic factor

$P(x) = x^3 + x^2 - 14x - 24$

1. Possible solutions for x are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6 \ldots$

We find that $p(-2) = 0$

2. $P(x) = (x+2)(ax^2 + bx + c)$

3. $x^3 + x^2 - 14x - 24$
$=$
$ax^3 + (b+2a)x^2 + (c+2b)x + 2c$
so $a = 1, b = -1, c = -12$

4. $p(x) = (x+2)(x^2 - x - 12)$
$= (x+2)(x-4)(x+3)$

Polynomials

Example

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R \setminus \{0\}$, then What are the factors

B

Let $p(x) = 4x^3 - 13x^2 - ax$.

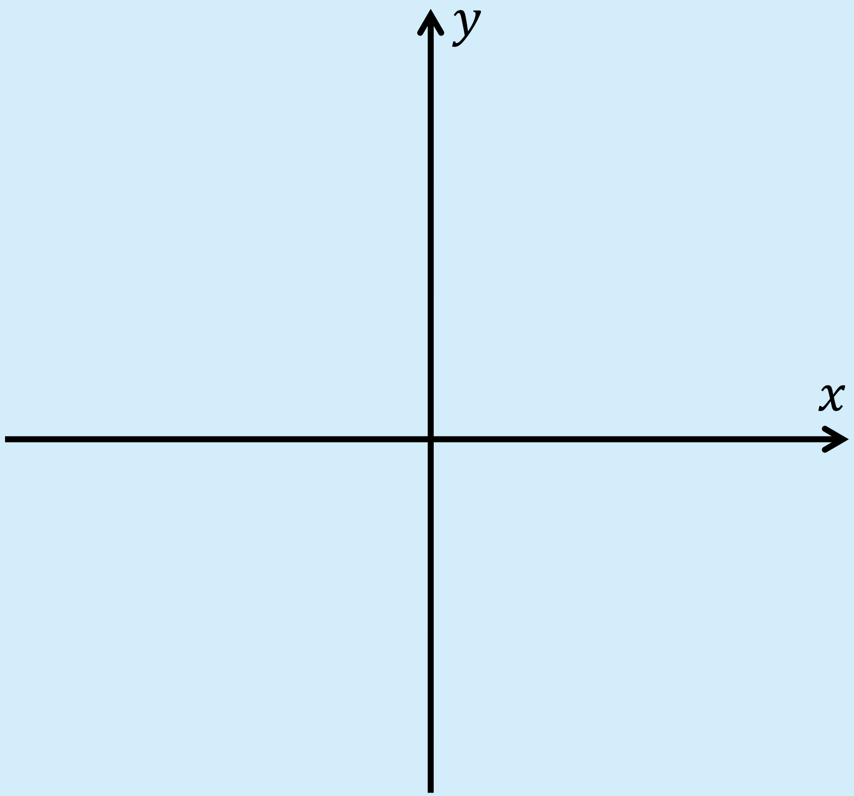
If $(x + a)$ is a factor, then

$$\begin{aligned} p(-a) &= 4(-a)^3 - 13(-a)^2 - a(-a) \\ &= -4a^3 - 12a^2 \\ &= -4a^2(a+3) = 0 \end{aligned}$$

Since $a \in R \setminus \{0\}$, $a = -3$.

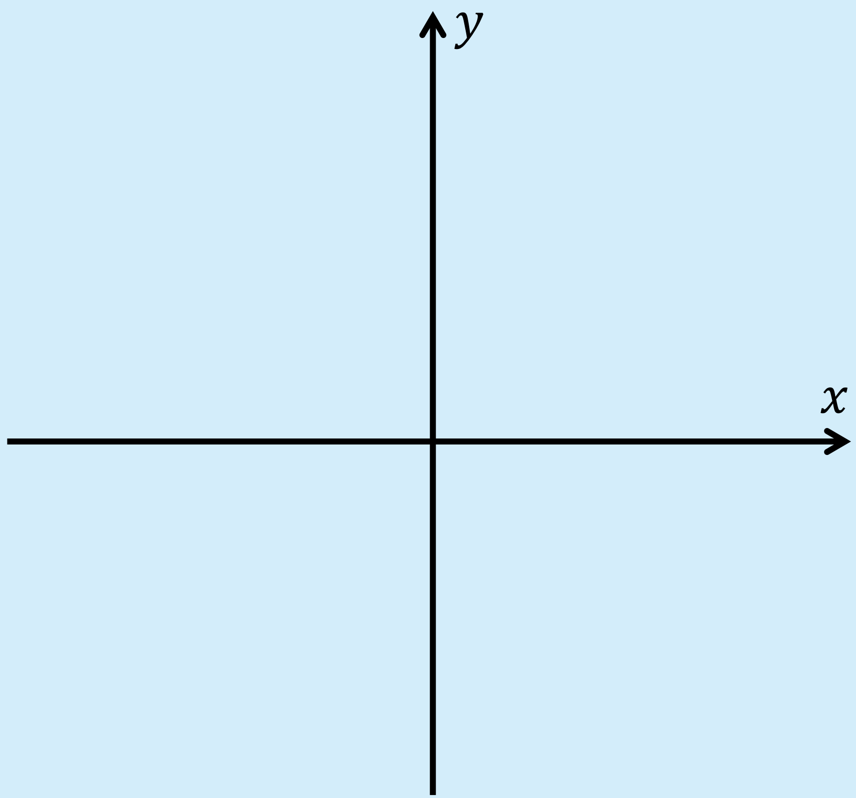
Sketch the graph of the following polynomials.

a. $y = (1 - x)(x + 2)^2$



Sketch the graph of the following polynomials.

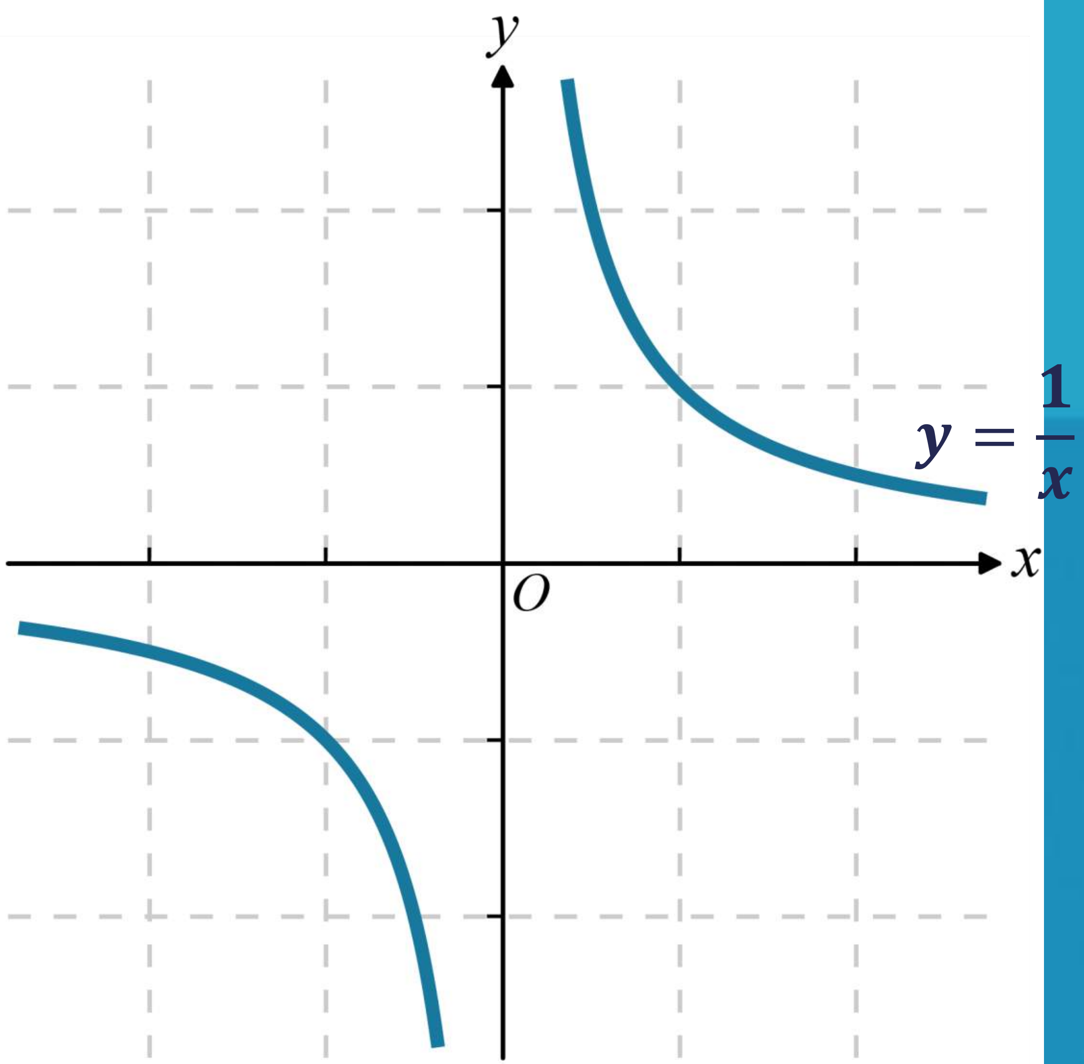
b. $y = (x+2)^3 - 8$



- A **hyperbola** is a graph of the form

$$y = \frac{a}{x-b} + c$$

- A hyperbola has:
  - a vertical asymptote at $x = b$
  - a horizontal asymptote at $y = c$



The process for short division:

- **'Short division'** is used to convert from $\boldsymbol{\frac{x+1}{x+2}} \rightarrow \boldsymbol{a+\frac{b}{x+2}}$

- 'More digestible' form of a hyperbola.

$$\frac{x+1}{x+2}$$

$$= \frac{\boldsymbol{x+2}-\boldsymbol{2}+1}{\boldsymbol{x+2}}$$

$$= \frac{\boldsymbol{x+2}}{\boldsymbol{x+2}} + \frac{-2+1}{x+2}$$

$$= \boldsymbol{1} + \frac{-1}{x+2}$$

Introduce $+\boldsymbol{2}$ so that the denominator's $\boldsymbol{x+2}$ is 'mimicked' in the numerator.

The $-\boldsymbol{2}$ is needed so that the fraction is unchanged overall.

It can now be split into two separate fractions. Because the denominator has been 'mimicked', one of these fractions will simplify.

Polynomials

Example

Express the fractions below in the form $a + \frac{b}{x+1}$, where a and b are integers.

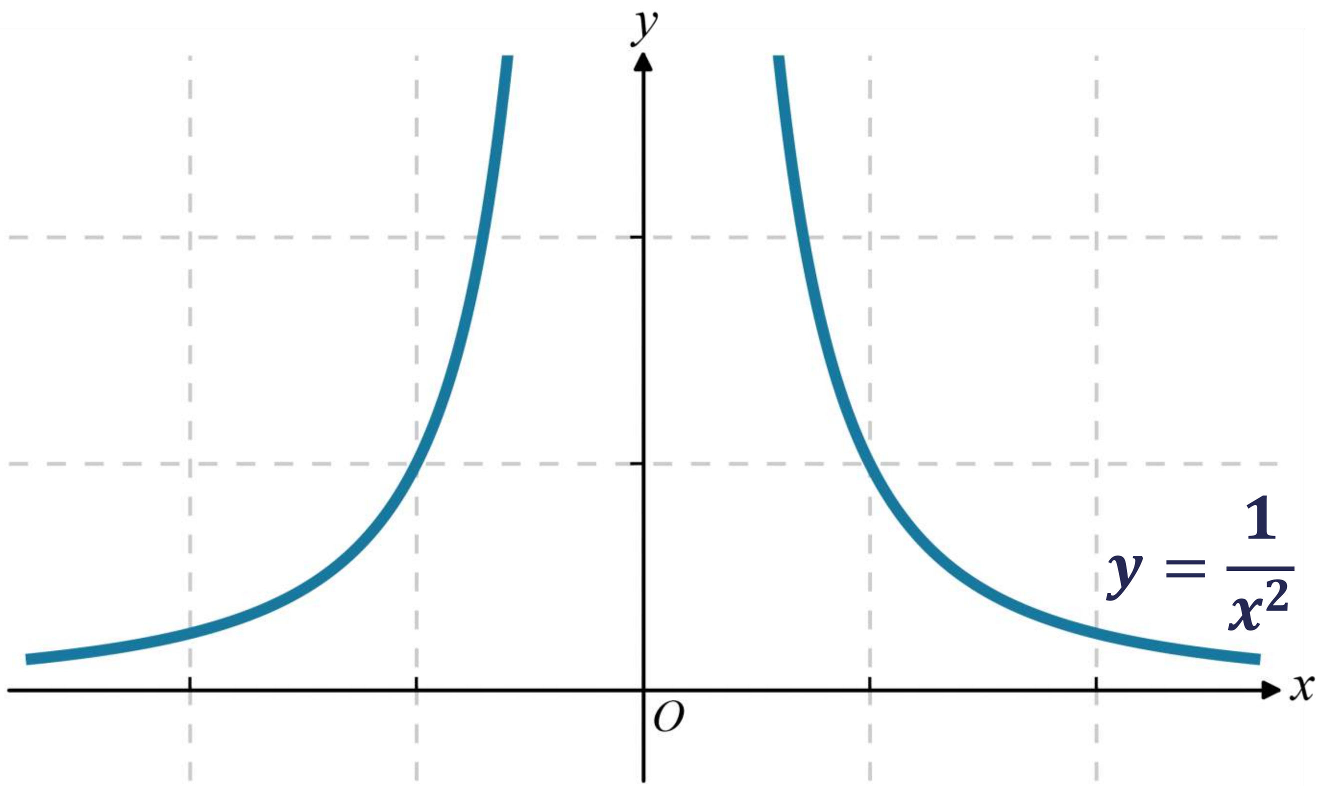
$$\frac{x-3}{x+1} \qquad\qquad \frac{2x-3}{x+1}$$

Polynomials

- A **truncus** has the form

$$y = \frac{a}{(x-b)^2} + c$$

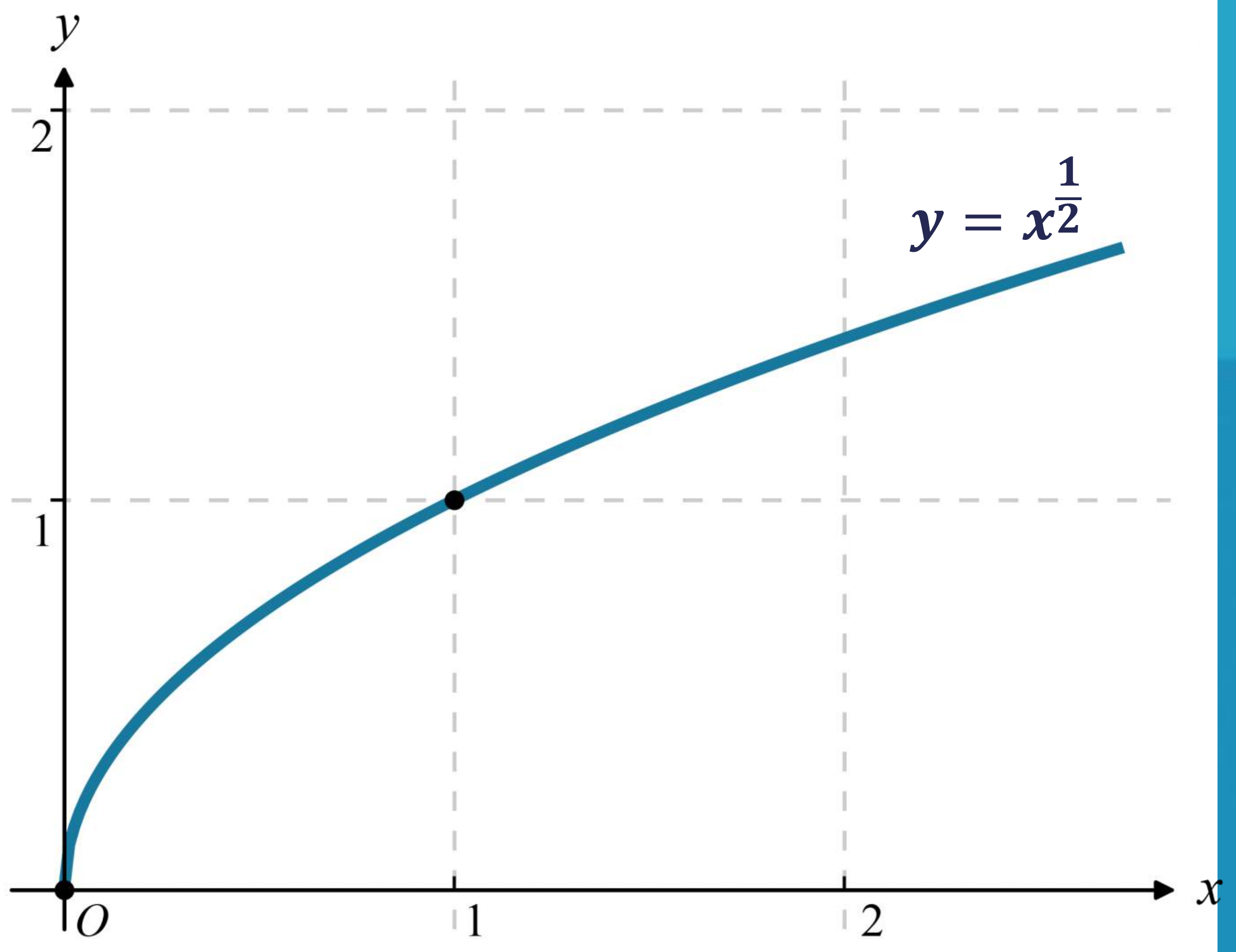
- Just like a hyperbola, a truncus has:
  - a vertical asymptote at $x = b$
  - a horizontal asymptote at $y = c$



Polynomials

- $y = \sqrt{x} = x^{\frac{1}{2}}$

- Note: All power functions where this general shape.

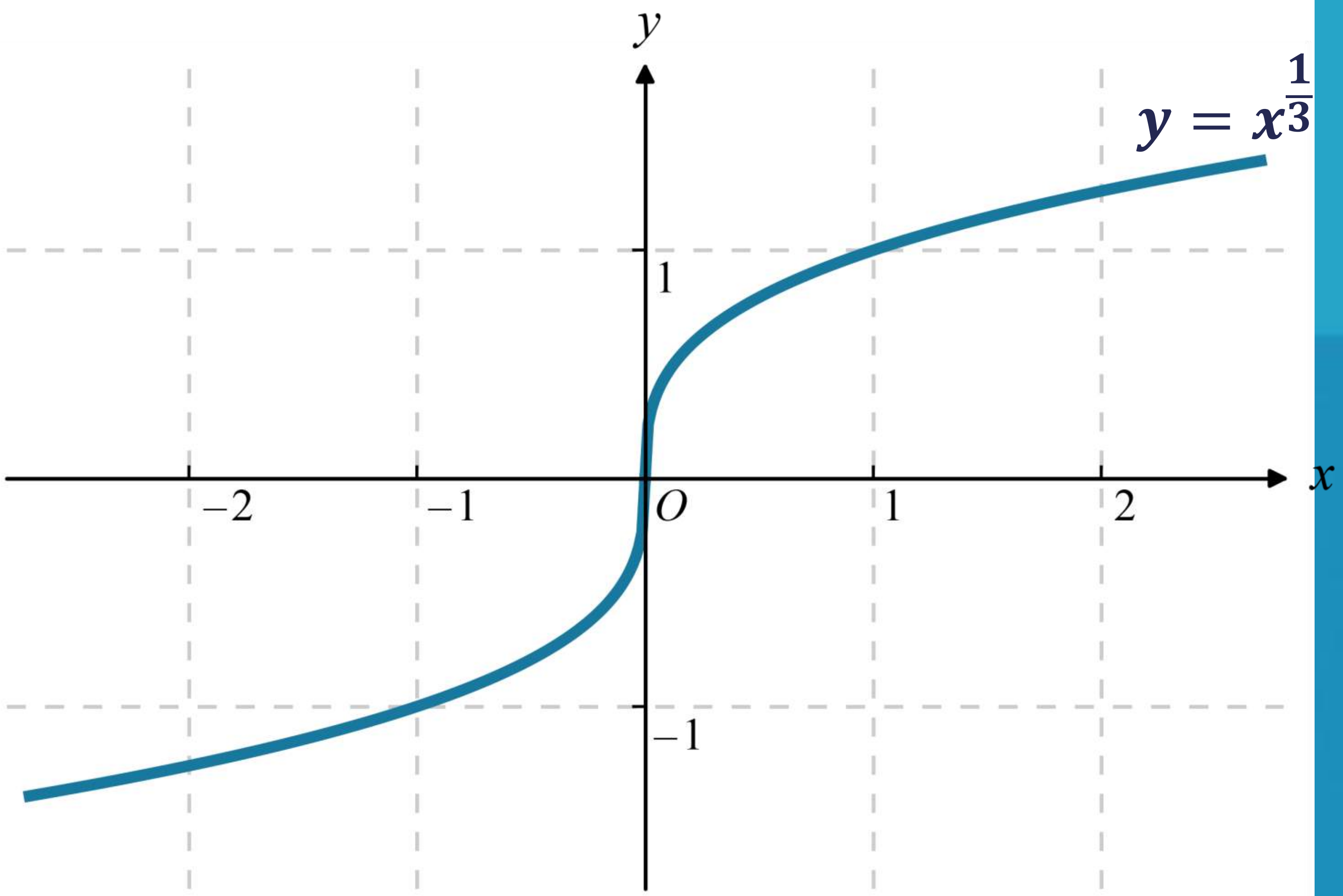


Polynomials

Power functions

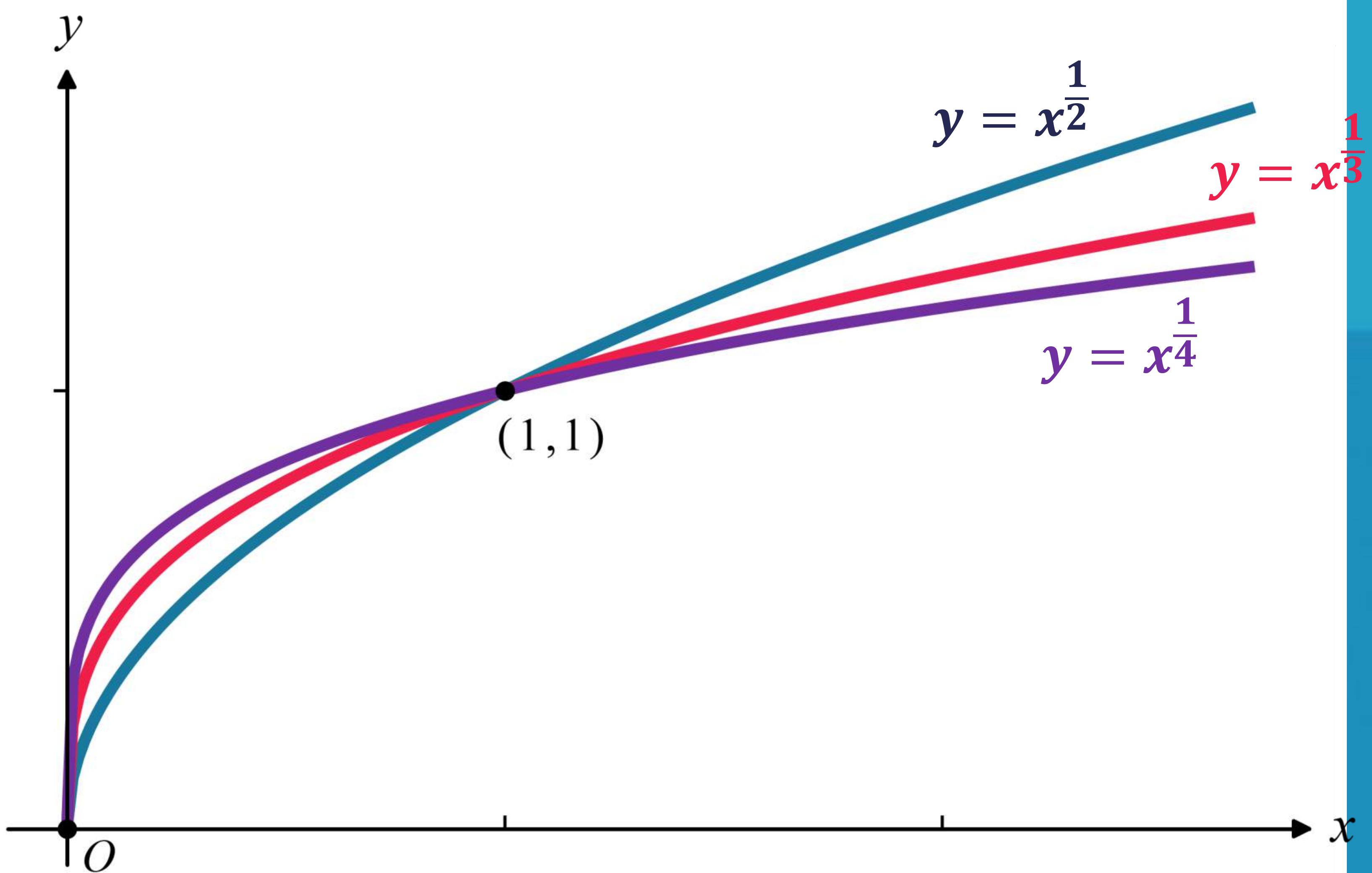
- $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

- Note: All power functions where $f(x) = x^{\frac{1}{a}}$ and a is an **odd** number follow this general shape.



Power functions

Graphs of $y = x^{\frac{1}{n}}$, where n is an integer, will pass through $(0,0)$ and $(1,1)$.



Functions

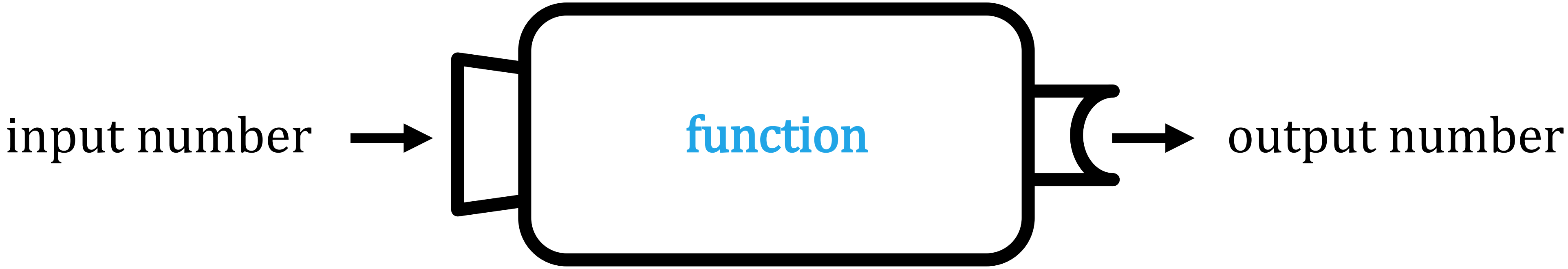
don't really need to worry
about these!
(aside from the next slide)

Relation: relationship between variables.

- Eg. $y^2 + x^2 = 1$ is a relation with the variables x and y.

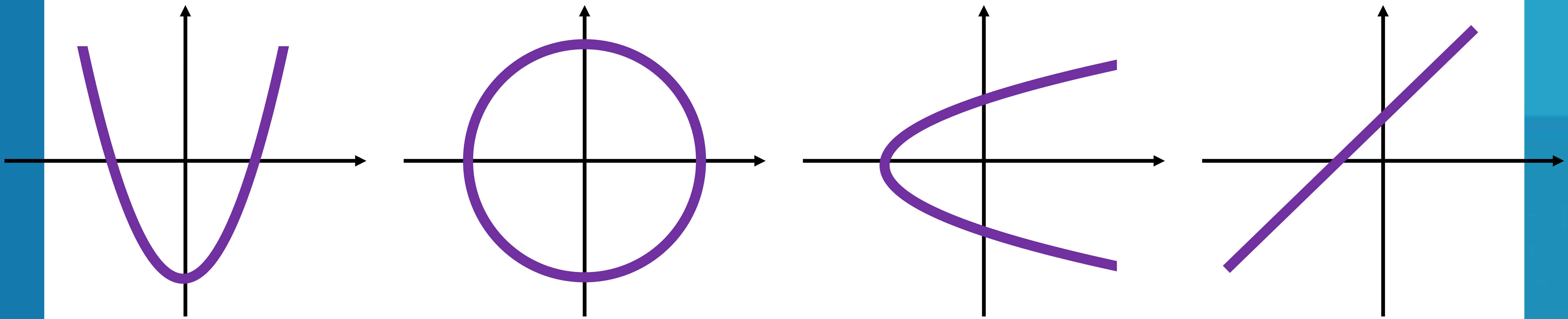
Function: specific relation.

- Only **ONE** y-value per x-value .
- You can think of a function like a machine:



Functions

Four types of relations:

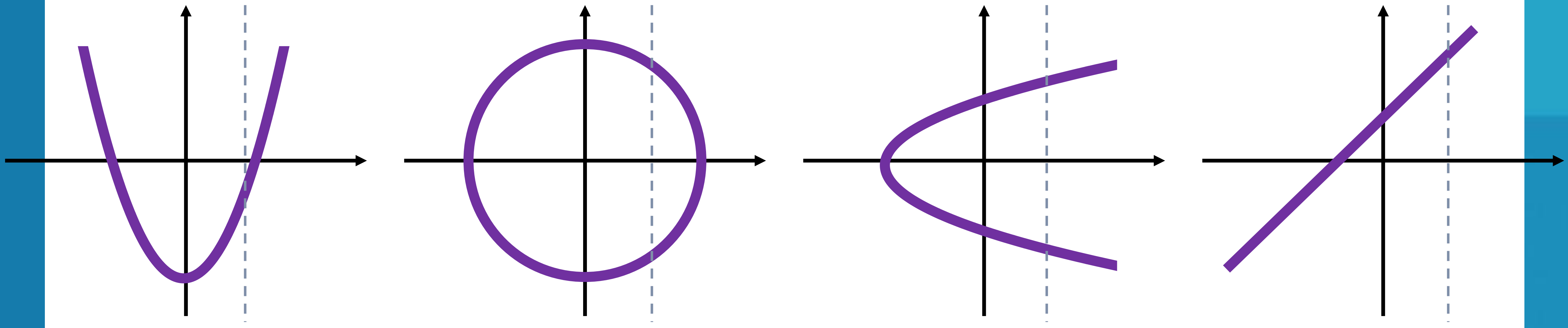


many – to – one | many – to – many | one – to – many | one – to – one

In order for a relation to be a function, ***only one x value can be related to each y value***
Which of the above relations are functions?

Functions

Four types of relations:



many – to – one

function

many – to – many

not function

one – to – many

not function

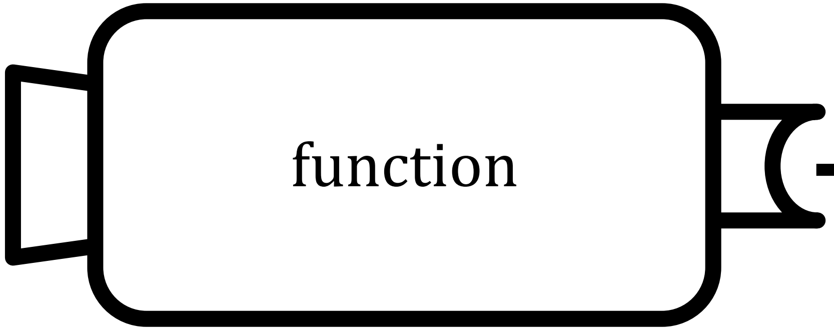
one – to – one

function

Vertical line test: if the line touches the relation more than once at any point, then it is not a function

- **Domain:** The x values you can put into the function
- The **input** of the function.

- **Range**: The y values you get out of the function.
- The **output** of the function.

domain (input) →  → **range** (output)

Functions

- Functions have a particular notation to describe them.

$$f : A \to B, f(x) = [\text{rule}]$$

The functions 'name'. f is commonly used, as are g and h.

This is the **domain** – the **input**: the x-values that CAN be plugged into the function.

Codomain: always R in Methods dw

The rule – the machine turning inputs into outputs.

Maximal domain

Domains may not be defined sometimes - assume it is the **maximal domain** if so.

- For polynomials, the maximal domain will be R.
- However, some numbers don't 'work' in the function so you need to exclude these from the domain.

$$\frac{1}{x} \qquad \sqrt{x} \qquad \log_e(x) \qquad \tan(x)$$

When there is a fraction with x in the denominator, **the denominator cannot be equal to zero**.

Whatever's under a square root **cannot be less than zero**.

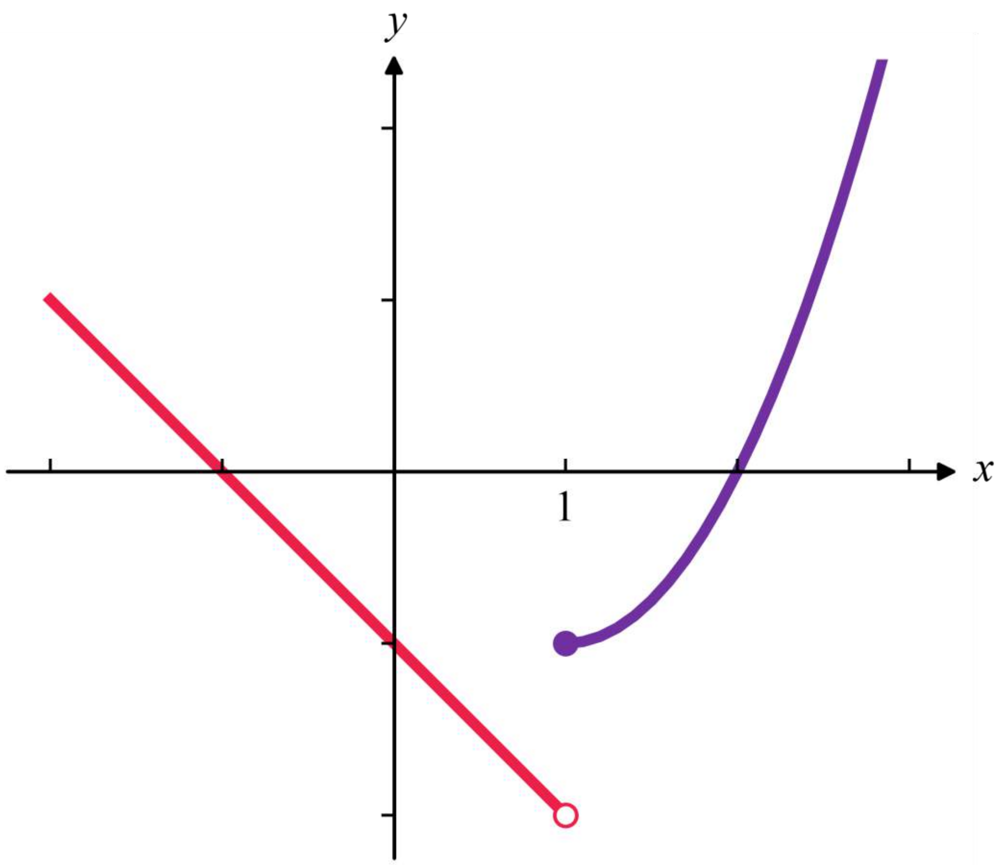
We'll look at these later in the year.

Hybrid/piecewise functions has **different rules over different x-values**.

- For example, the hybrid function

$$f(x) = \begin{cases} -x - 1 & x < 1 \\ (x-1)^2 - 1 & x \geq 1 \end{cases}$$

is shown on the right.



Functions

Example

Let the hybrid function f be defined

$$f(x) = \begin{cases} (x+2)^2 - 2 & x < 0 \\ x^2 + bx + c & x \geq 0 \end{cases}$$

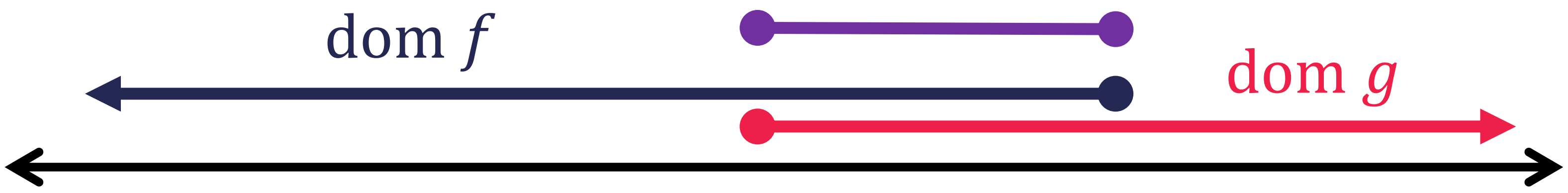
If $f(x) = f(-x)$ for all x, find the values of b and c.

- **Sum functions** are the sum of two functions:

$$h(x) = f(x) + g(x)$$

- The domain of $h(x)$ **will only exist** if both:
  - $f(x)$ **is defined**
  - $g(x)$ **is defined**

$$\text{dom } h = \text{dom } f \cap \text{dom } g$$



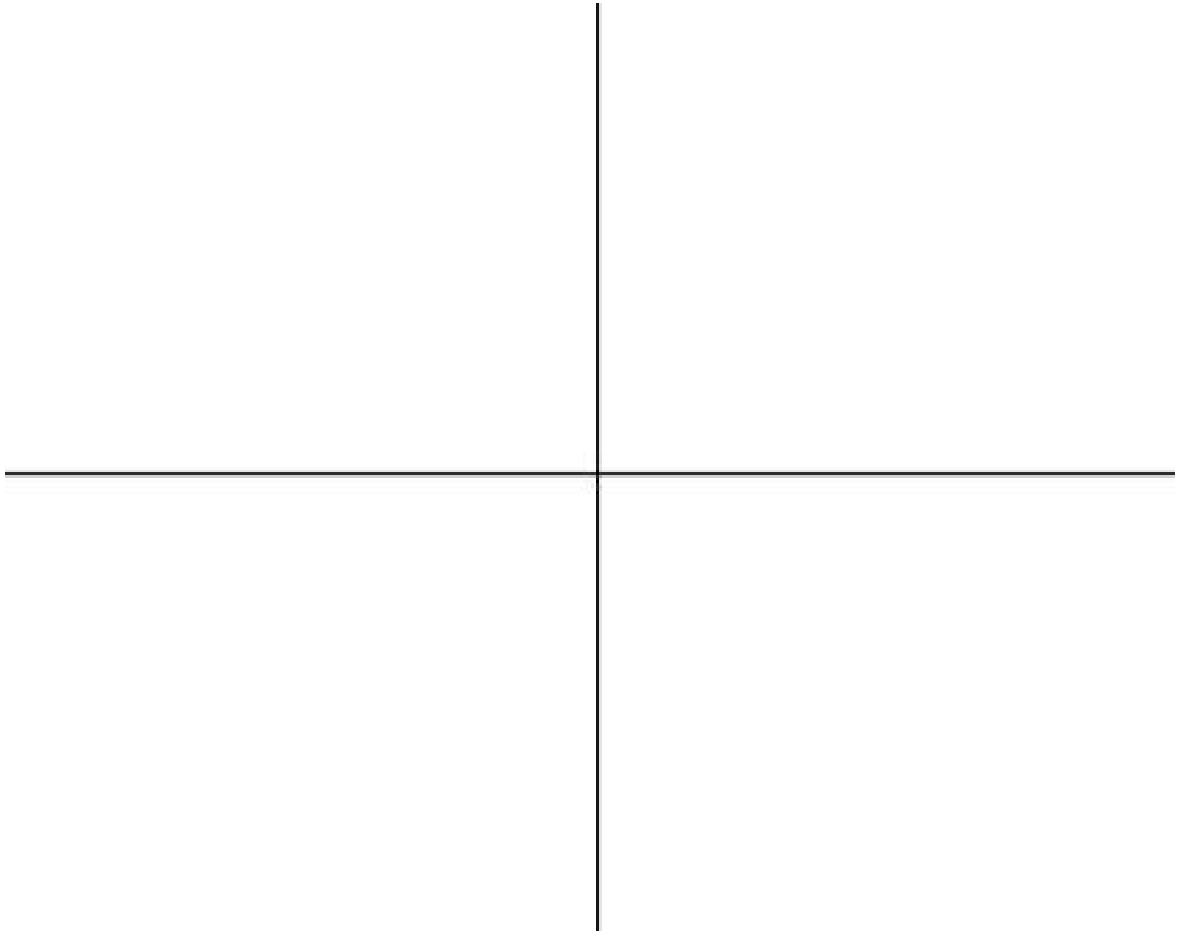
Functions

Addition of Ordinates

To sketch a sum function:

1. Faintly sketch both functions
2. Look for where one graph is 0 (keep it)
3. Look for where they intersect (double it)

$$x^2 - 1 + \frac{1}{x+2}$$



Functions

Example

CAS allowed

If $f: (-\infty, 2] \to R$, $f(x) = \sqrt{2-x} + 6$ and $g: (-2, \infty) \to R$, $g(x) = \frac{4}{(x+2)^2}$, then the maximal domain of the function $f + g$ is

A. R

B. $[-2, 2)$

C. $(-\infty, -2)$

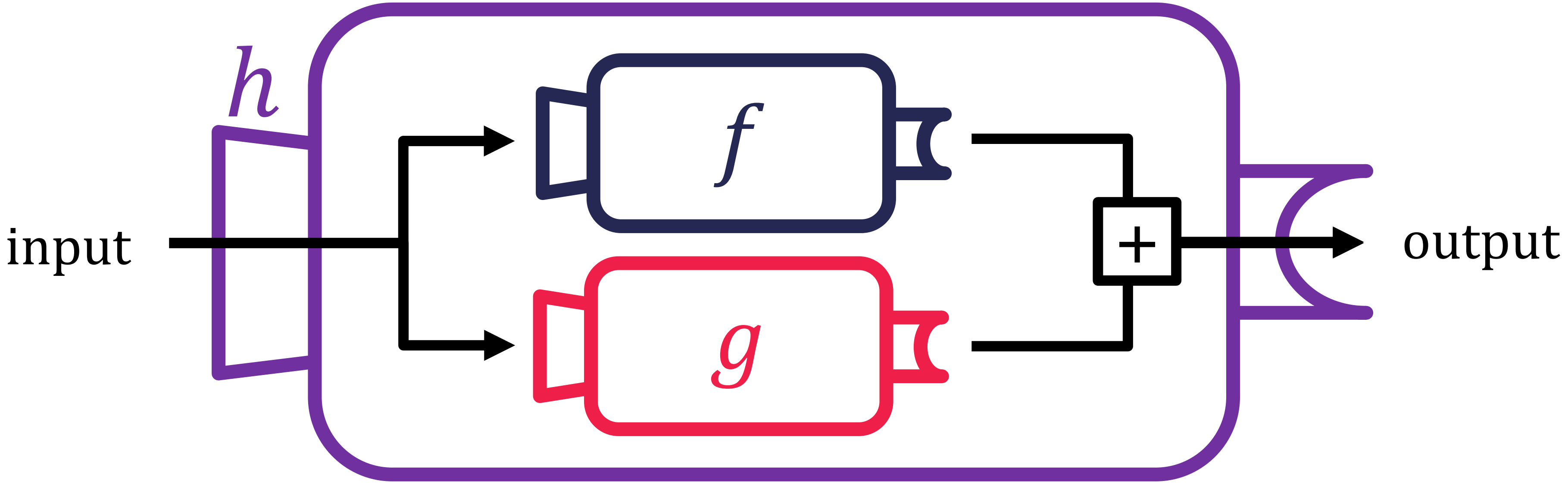
D. $(-2, 2]$

E. $[2, \infty)$

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$$

$$\operatorname{dom} h = \operatorname{dom} f \cap \operatorname{dom} g$$

Remember *why* this is the case.



Composite functions

- A **composite function** has **one function inside another function**.

$$h(x) = f\big(g(x)\big)$$

Composite function

Outside function

Inside function

Can also be written like this

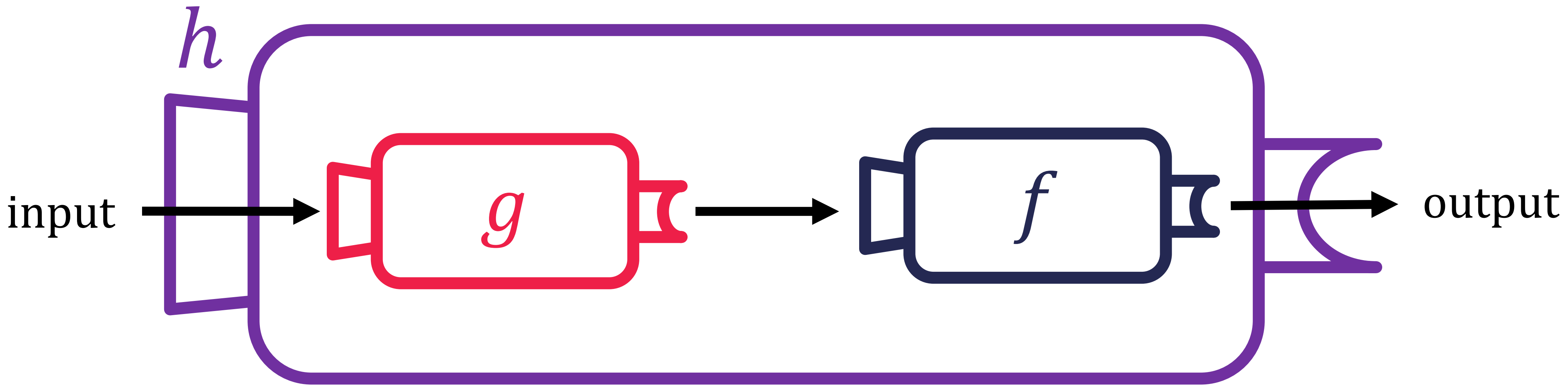
$f\big(g(x)\big) = (f \circ g)(x)$

$g\big(f(x)\big) = (g \circ f)(x)$

- Numbers in $\boldsymbol{h}$ are first being processed by $\boldsymbol{g(inside)}$ and then by $\boldsymbol{f(outside)}$.

- Functions are machines.
- **Composite functions** are machines which have two machines inside them.

$$h(x) = f\big(g(x)\big)$$

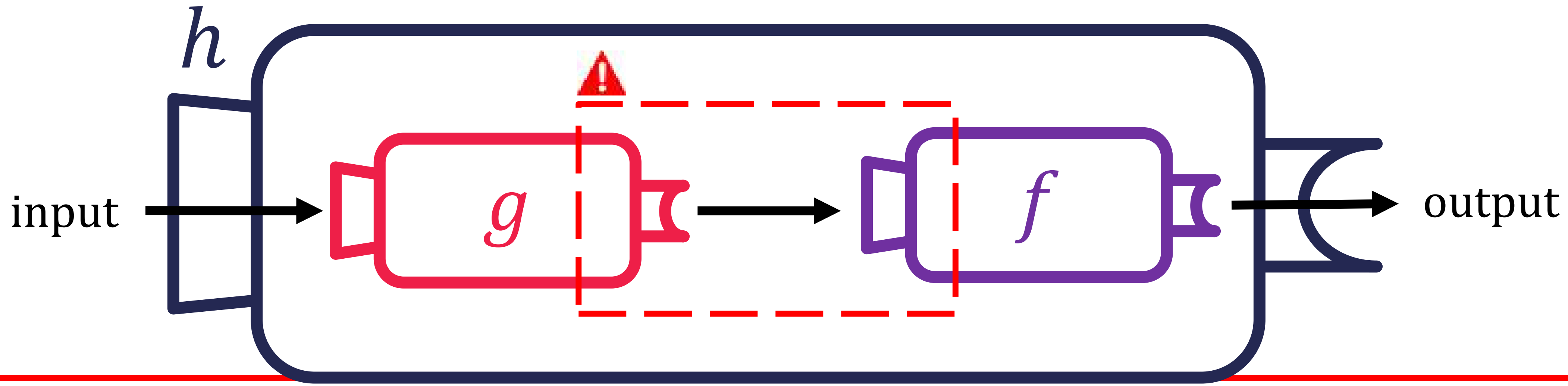


IMPORTANT: **numbers coming out of g are going into f**. For h to work, numbers out of g must be able to go into function f.

In other words, **the range of g must be a subset of the domain of f**

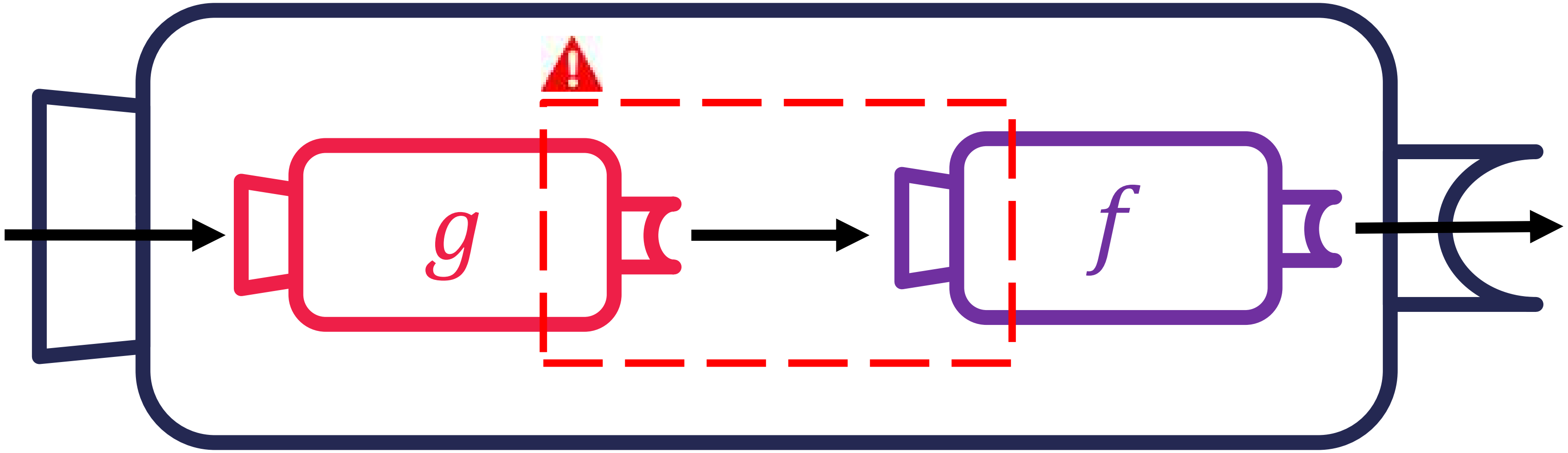
RIDO – Range Inner is Subset of Domain Outer

domain of composite $f(g(x))$ = domain of $g(x)$.



Restricting Composite (HELL TOPIC)

- Sometimes, a variable may change the domain or range of g.
- $f\big(g(x)\big)$ may or may not be defined when that variable takes different values.
- Often the restriction will be the domain of f. For $f\big(g(x)\big)$ to be defined, the variable must take a value such that what's coming out of g can go into f.



HELL TOPIC in 5 EASY STEPS

For f o g:

1. RIDO (range inner is smaller/equal to domain of outer) $(ran\ g \subseteq dom\ f)$
2. If not: Restrict g range to equal f domain $(at\ most, ran\ g = dom\ f)$
3. Draw g equation
4. Highlight parts of g range that are in f domain
5. Define as new domain

Functions

Example

For $f(x) = x^2 - 1, x \in R$ and $g(x) = \sqrt{x}, x \geq 0$

Show that $g \circ f$ is not defined and make a restriction f* so it is defined

	Domain	Range
f	(-∞,∞)	[-1,∞)
g	[0,∞)	[0,∞)

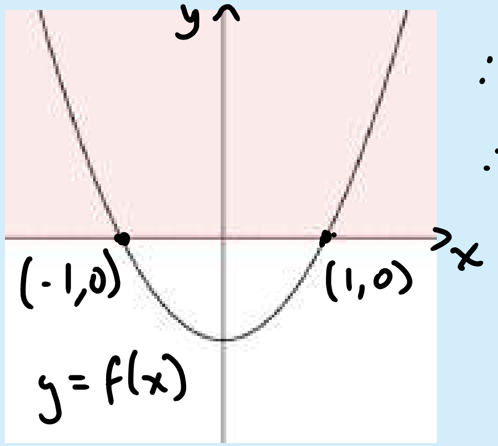
for gof to be defined, we require ran $f \subseteq$ dom g

but ran $f = (-\infty, \infty)$ and dom $g = [0,\infty)$.

$\therefore$ ran $f \not\subseteq$ dom g

$\therefore$ gof is undefined.

at most, ran $f^* =$ dom g



$\therefore$ ran $f^* = [0,\infty)$

$\therefore$ dom $f^* = [1,\infty)$

Inverse Functions

Essentially, **inverse functions** 'reverse' another function: $f^{-1}(f(x)) = x$

<u>Key Properties:</u>

- You swap x and y variables
  - You're basically putting the y as the x and the x as the y
- Because of this: the domain and range of f(x) will be swapped
  - dom $f^{-1}(x)$ = ran f(x)
  - ran $f^{-1}(x)$ = dom f(x)
- Graphically it is reflected on the y=x line
- <u>MUST BE ONE-TO-ONE FUNCTION</u>

How to find the rule for an inverse function

The inverse of f is f^{-1} and MUST BE ONE TO ONE (horizontal AND vertical tests)

1. Write **"swap $\boldsymbol{x}$ and $\boldsymbol{y}$"** (this COUNTS as working)
2. Rewrite function with **$\boldsymbol{x}$ and $\boldsymbol{y}$ swapped**
3. Rearrange make y the subject
4. Write "$\boldsymbol{y = f^{-1}(x)} = \ldots$"

Functions

Example

VCAA 2012

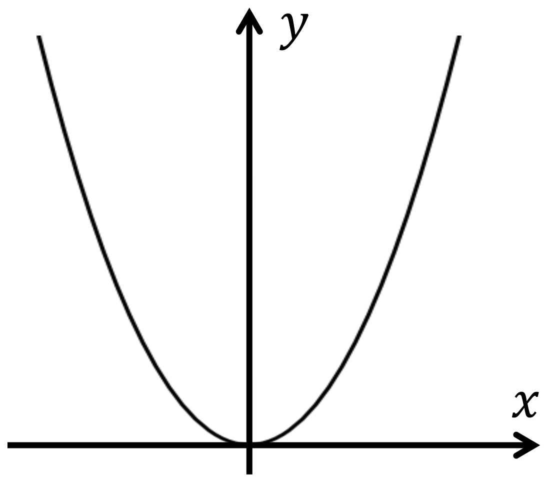
The rule for function h is $h(x) = 2x^3 + 1$. Find the rule for the inverse function h^{-1}.

One-to-one functions

- If a function is not one-to-one, we can **restrict the domain** to force it.

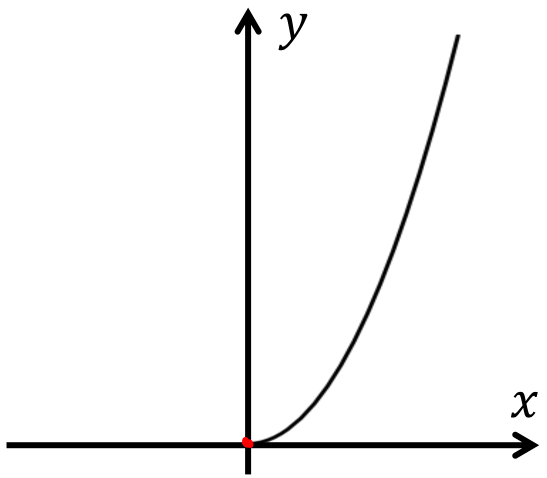
$f: R \to R,\ f(x) = x^2$

not one-to-one



Domain restricted to $[0, \infty)$

one-to-one



Question 3

Marks	0	1	2	Average
%	9	34	56	1.5

$h(x) = 2x^3 + 1$, let $y = 2x^3 + 1$

for inverse, swap x and y

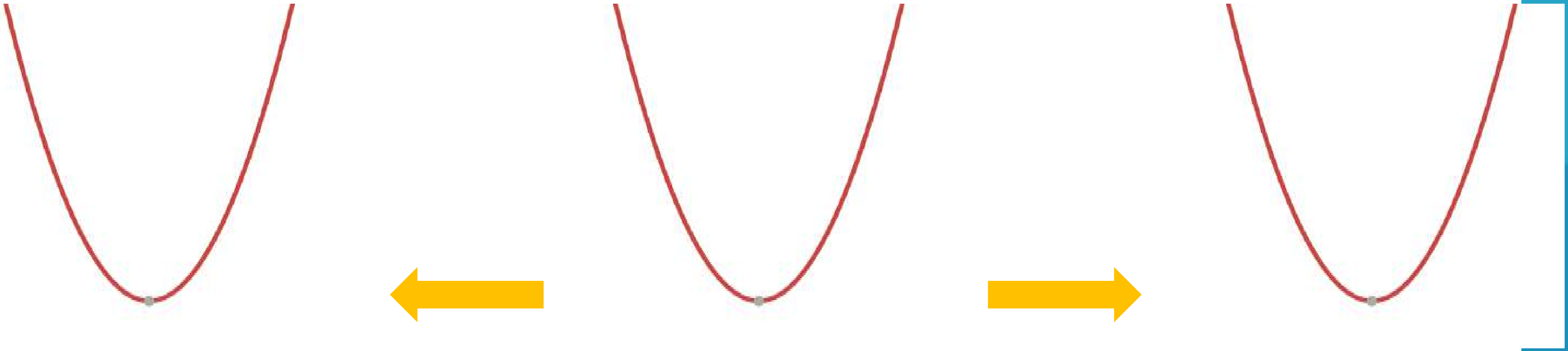
$\Rightarrow x = 2y^3 + 1$, make y the subject

$y = \sqrt[3]{\frac{x-1}{2}}$, y is the inverse of $h(x)$

$h^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$, $x \in R$

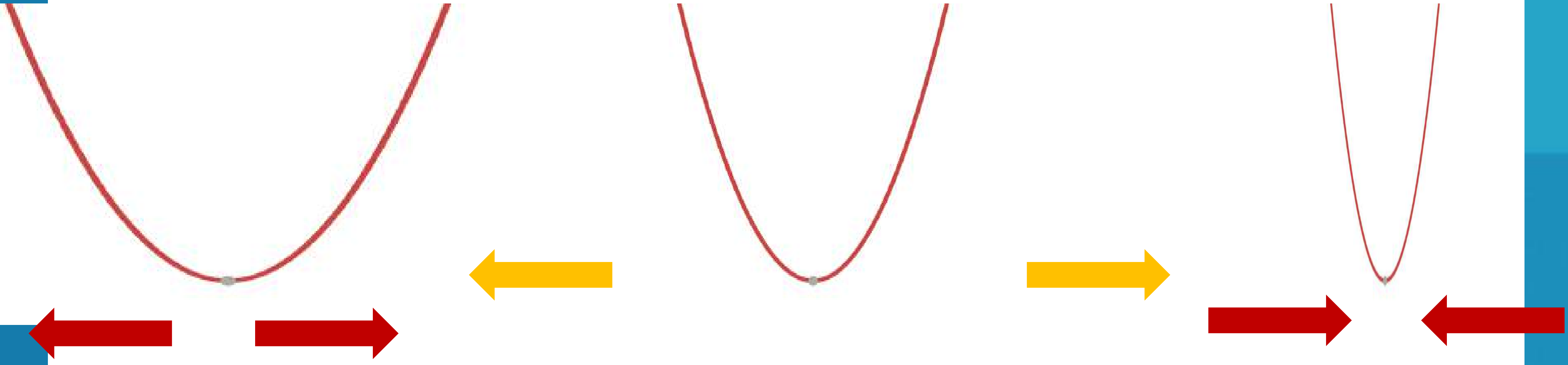
It is important that students do not proceed directly from $y = 2x^3 + 1$ to $x = 2y^3 + 1$. This is not correct working. Students need to indicate that new working is starting. An approach is as shown above. A common error was to write $h^{-1}(x) = x = 2y^3 + 1$.

- There are three types of transformations:



DILATIONS

- There are three types of transformations:



DILATIONS

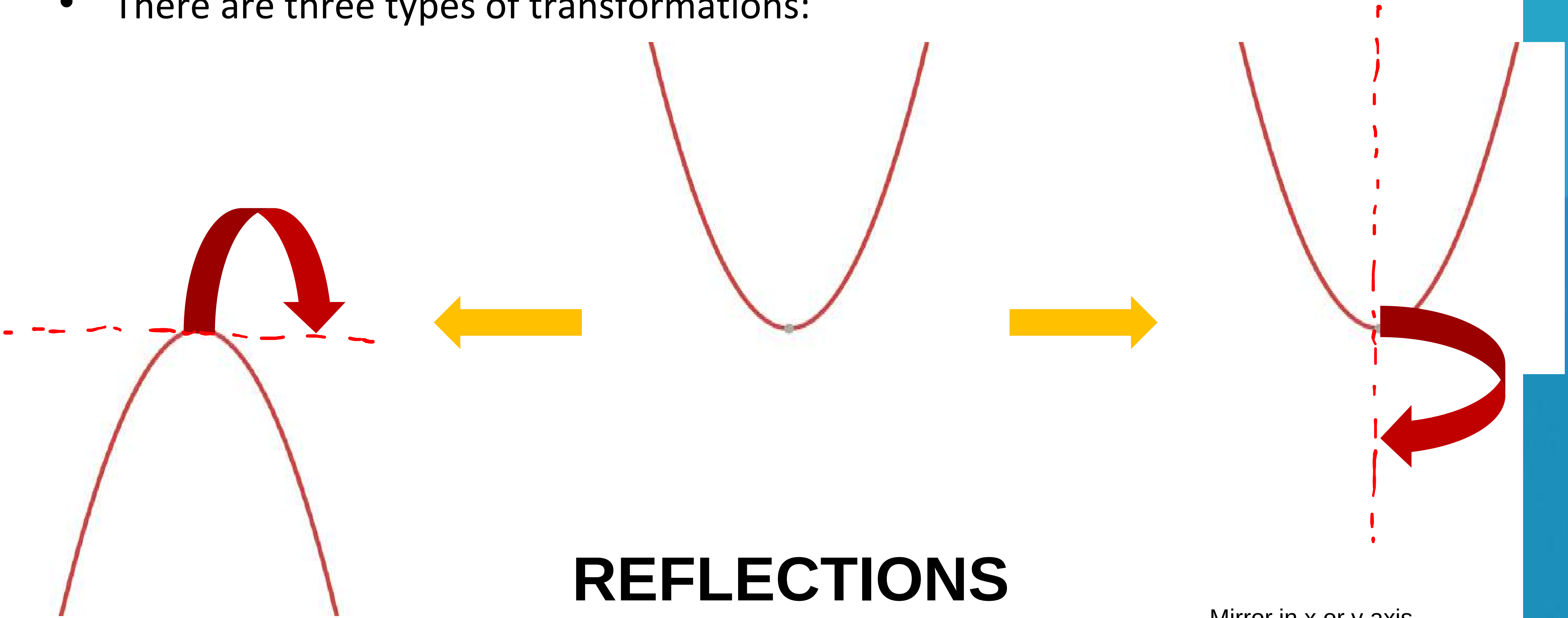
Squishing in or out

- There are three types of transformations:

REFLECTIONS

Types of transformations

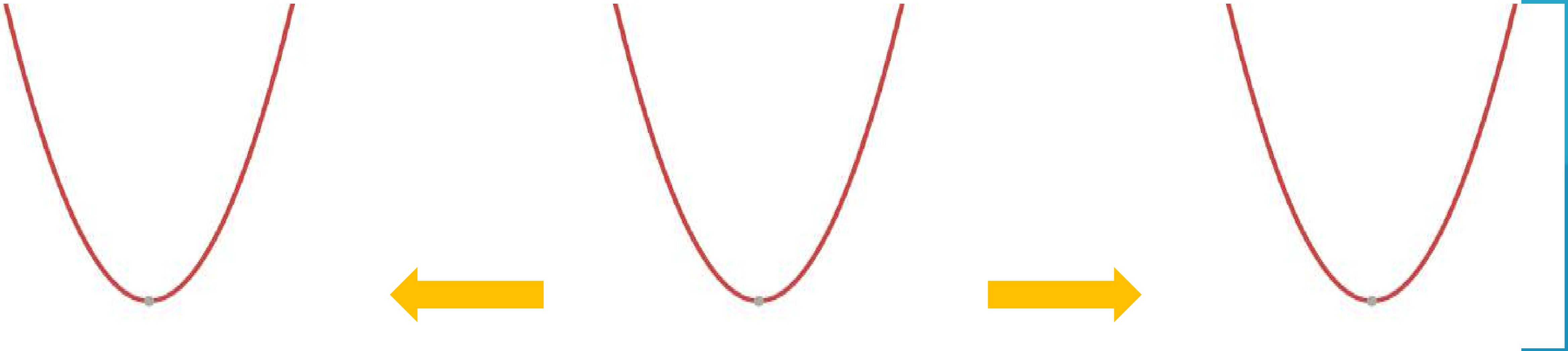
- There are three types of transformations:



REFLECTIONS

Mirror in x or y axis

- There are three types of transformations:

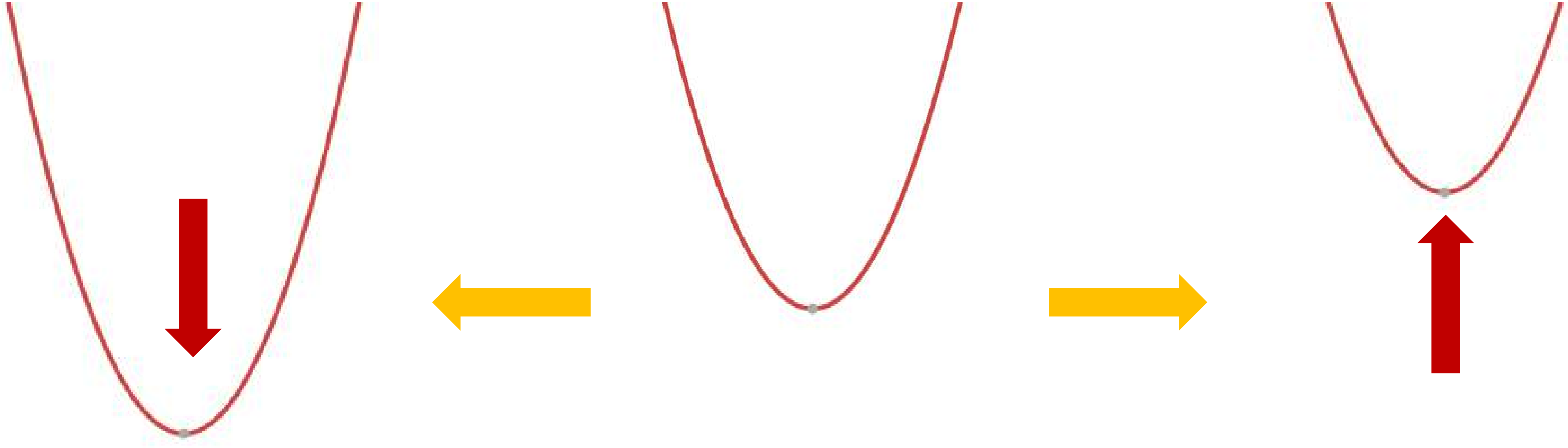


TRANSLATIONS

Transformations

Types of transformations

- There are three types of transformations:



TRANSLATIONS

Move in any direction

Types of transformations

Dilation by factor a **from the y axis** OR **in the direction of the x axis**	affects x values
Dilation by factor b **from the x axis** OR **in the direction of the y axis**	affects y values
Reflection in the y axis	affects x values
Reflection in the x axis	affects y values
Translation c in the pos/neg y direction OR c up/down	affects y values
Translation d in the pos/neg x direction OR d right/left	affects x values

Unless ***specifically*** stated, transformations should be done in this order:

DILATIONS

REFLECTIONS

TRANSLATIONS

- Why is this so?

If dilations or reflections are done after translations, then the translations are altered

Coordinate method:

- Takes longer, less room for errors (use in beginning)
- Eg. Dilation a from y axis, b from x axis, reflection in x and y axes, translation c up and d right

$$(x, y) \rightarrow (ax, by) \rightarrow (-ax, -by) \rightarrow (\underline{-ax + d}, \underline{-by + c})$$

(x', y') denotes the 'new' or transformed coordinate

$$\boxed{x'} = -ax + d$$
$$\boxed{y'} = -by + c$$

$$x = -\frac{1}{a}(x' - d) \qquad y = -\frac{1}{b}(y' - c)$$

$$x = -\frac{1}{a}(x' - d) \qquad y = -\frac{1}{b}(y' - c)$$

From here the above equations can be substituted into the function

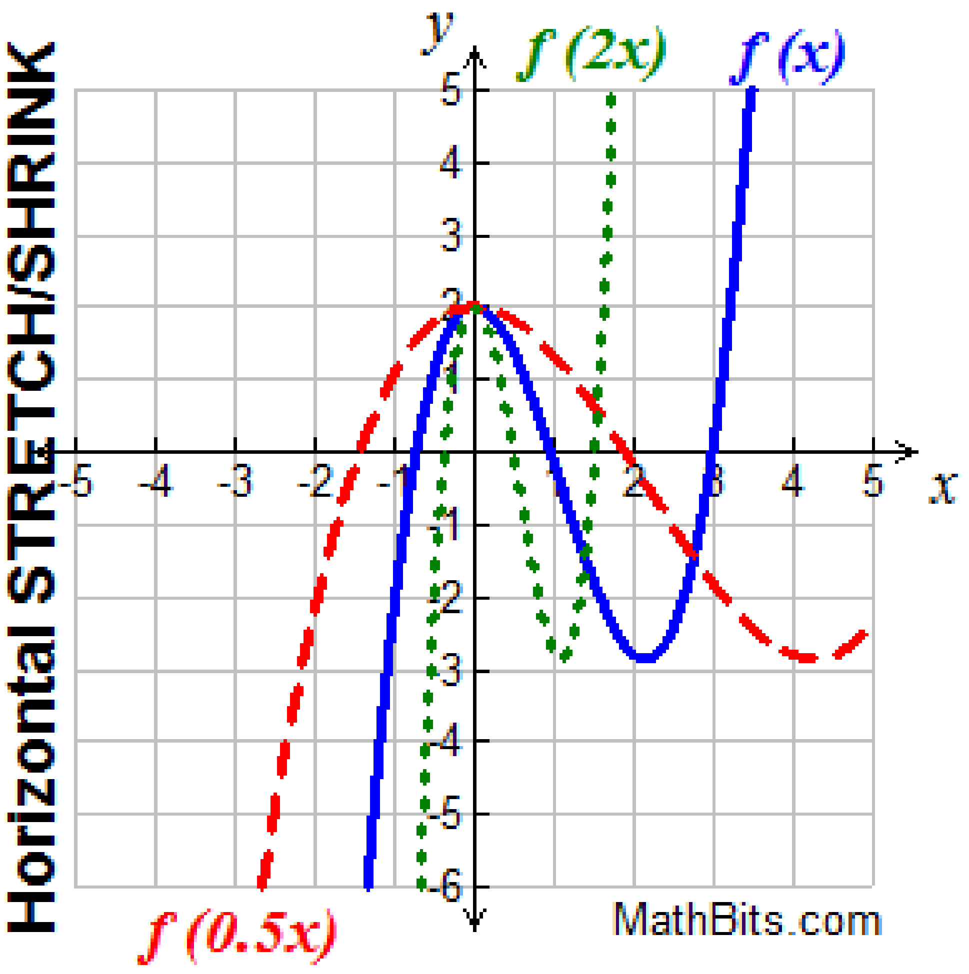
$$-\frac{1}{b}(y' - c) = f\left(-\frac{1}{a}(x' - d)\right)$$

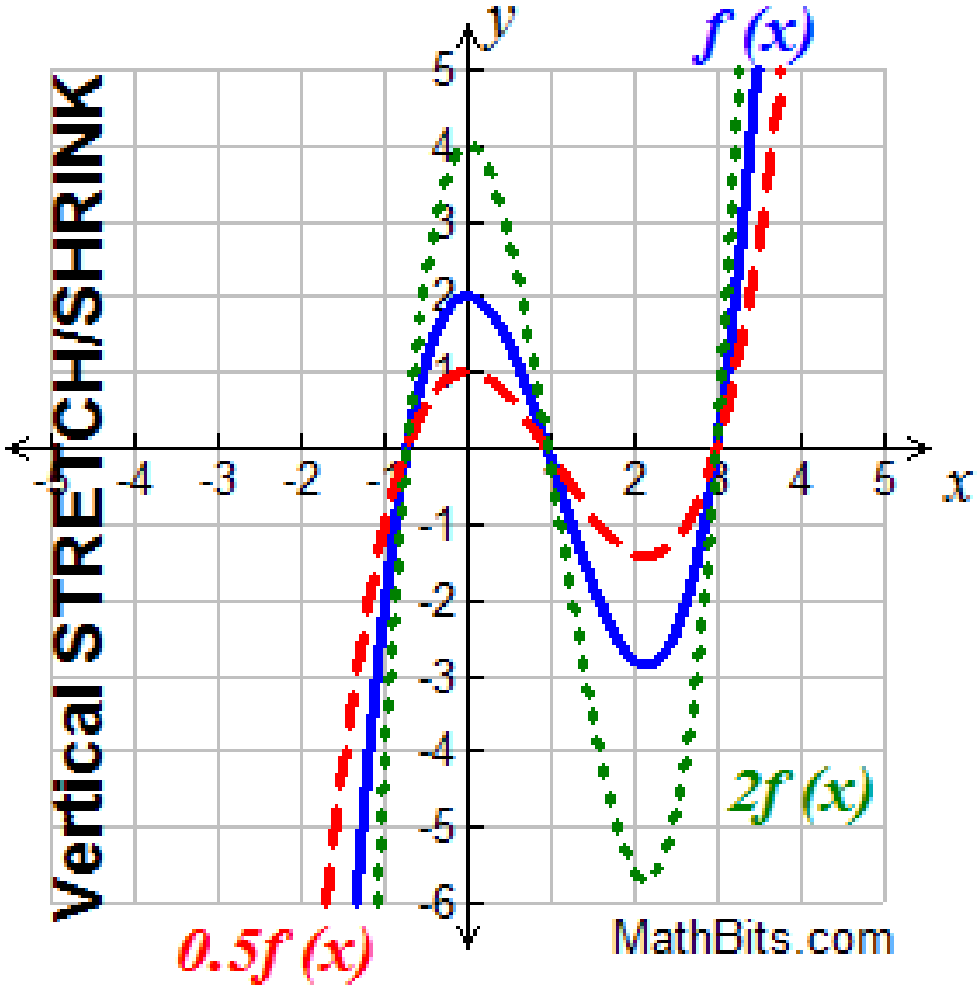
This can explain the next slide--->

Coordinate Steps

1. Convert (x,y) into (ax+b, cy+d)
2. Write “Let x’ = ax+b, y’=cy+d”
3. Rearrange so that x and y are subject
4. Sub into the function

Dilations in case

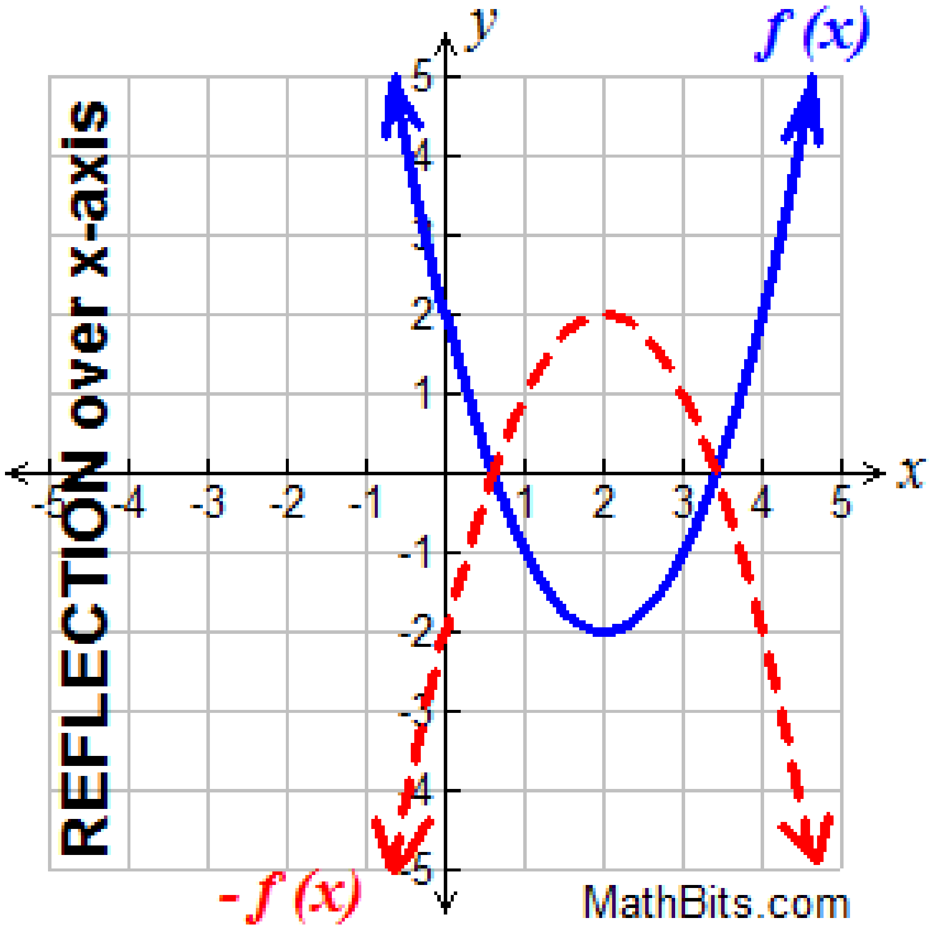


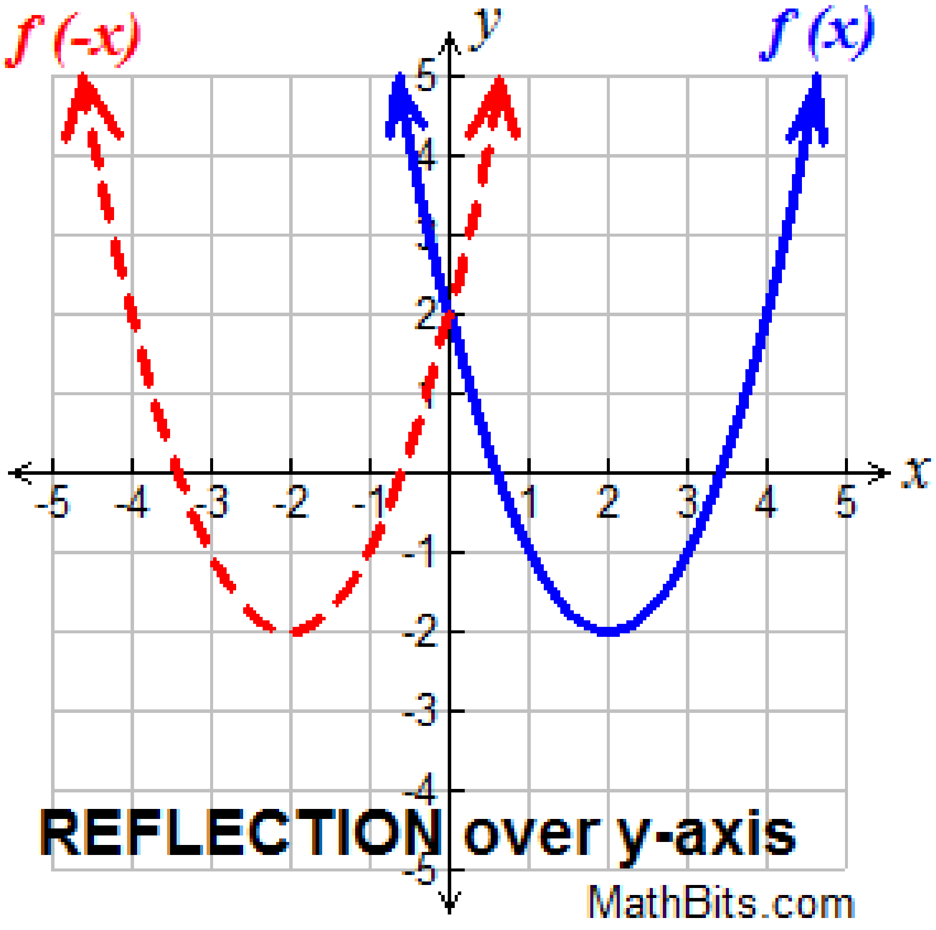


Dilation of factor in y axis = changes x only

Dilation of factor in x axis = changes y only

Reflections in case





Reflection x axis = mirror

Reflection y axis = mirror

Transformations

Example

A function $f(x) = x^2 + 3x$ undergoes the following transformations

- Dilation 3 from y axis and 5 from x axis
- Reflection in the y axis
- Translation 4 down and 1 right

Using the coordinate method, state the transformed function g(x)

1. Convert (x,y) into (ax+b, cy+d)
2. Write "Let x' = ax+b, y'=cy+d"
3. Rearrange so that x and y are subject
4. Sub into the function

Finding transformations

- To find what transformations to change one function to another, we do coordinate method backwards

- Begin by rearranging your new function so that parts can be equated:

$$y' = \frac{1}{4}(x' - 2)^2 \qquad y = x^2$$

$$\boxed{4y'} = \boxed{(x' - 2)}^2 \qquad \boxed{y} = \boxed{x}^2$$

$$y = 4y' \Rightarrow y' = \frac{y}{4} \qquad x = x' - 2 \Rightarrow x' = x + 2$$

$$y = 4y' \Rightarrow y' = \frac{y}{4} \qquad x = x' - 2 \Rightarrow x' = x + 2$$

So there is a dilation $\frac{1}{4}$ from the x axis and a translation 2 right

TIPS

Identify at the start which function is the original and which is the transformed

There will be multiple correct answers due to equivalent transformations

Finding Transformation Steps

1. Equate parts of the equation to transform

   Basically, the x parts need to match forms: eg. x^2 and $(x+b)^2$

2. Let y = y' transformations and x = x' transformations
3. Rearrange so y' and x' is subject
4. Write (x,y) ---> (x',y')
5. Follow DRT to describe the transformations

Transformations

THE PRO WAY!!!!!!

$$y = af\big(n(x-b)\big) + c$$

- Dilation by factor of a from x axis
- If a is negative it is x axis reflection

- Dilation by factor of $\frac{1}{n}$ from y axis
- Negative n is reflection in y axis

- Translation of b units right
- Negative b is left

- Translation of c units up
- Negative c is down

- **Always** write into this form to explain transformations!!!!!!!!!!!
- Coefficient of x HAS to be 1
- Order of listing transformations is DR T
  - Dilations, Reflections then Transformations

CAS allowed VCAA 2012 Exam 2

Let $f: R\backslash\{2\} \to R, f(x) = \frac{1}{2x-4} + 3$

A sequence of transformations map the graph of f to the graph of the function $g: R\backslash\{0\}, g(x) = \frac{1}{x}$. What is this sequence?

Q2e $f(x)=\frac{1}{2x-4}+3 \to g(x)=\frac{1}{x}$

The required transformation sequence is:

(1) Dilate $f(x)$ by a factor of 2 horizontally,

i.e. $f(x) \to f\left(\frac{x}{2}\right)=\frac{1}{x-4}+3$

(2) Translate to the left by 4 units,

i.e. $\to f\left(\frac{x+4}{2}\right)=\frac{1}{x}+3$

(3) Translate downwards by 3 units,

i.e. $\to f\left(\frac{x+4}{2}\right)-3=\frac{1}{x}=g(x)$

$\therefore\ a=2,\ c=-4$ and $d=-3$

Exponentials

An exponential function takes the form

$$y = a^x$$

- a is the base, and is any positive number except 1 (i.e. $a \in \mathbb{R}^+\backslash\{1\}$)
  - This means it can also be a fraction, e.g. $\frac{1}{2}$ - we'll look at the implications of this soon
- In methods, the base is commonly Euler's number, e
  - Useful to remember a rough decimal; equal to 2.718…
  - An actual number, not a variable to be solved for (treat it like π)

Index laws

$$\boldsymbol{a}^m \times \boldsymbol{a}^n = \boldsymbol{a}^{m+n}$$

$$\frac{\boldsymbol{a}^m}{\boldsymbol{a}^n} = \boldsymbol{a}^{m-n}$$

$$(\boldsymbol{a}^m)^n = \boldsymbol{a}^{m\times n}$$

$$\boldsymbol{a}^0 = 1$$

$$\boldsymbol{a}^{-m} = \frac{1}{\boldsymbol{a}^m}$$

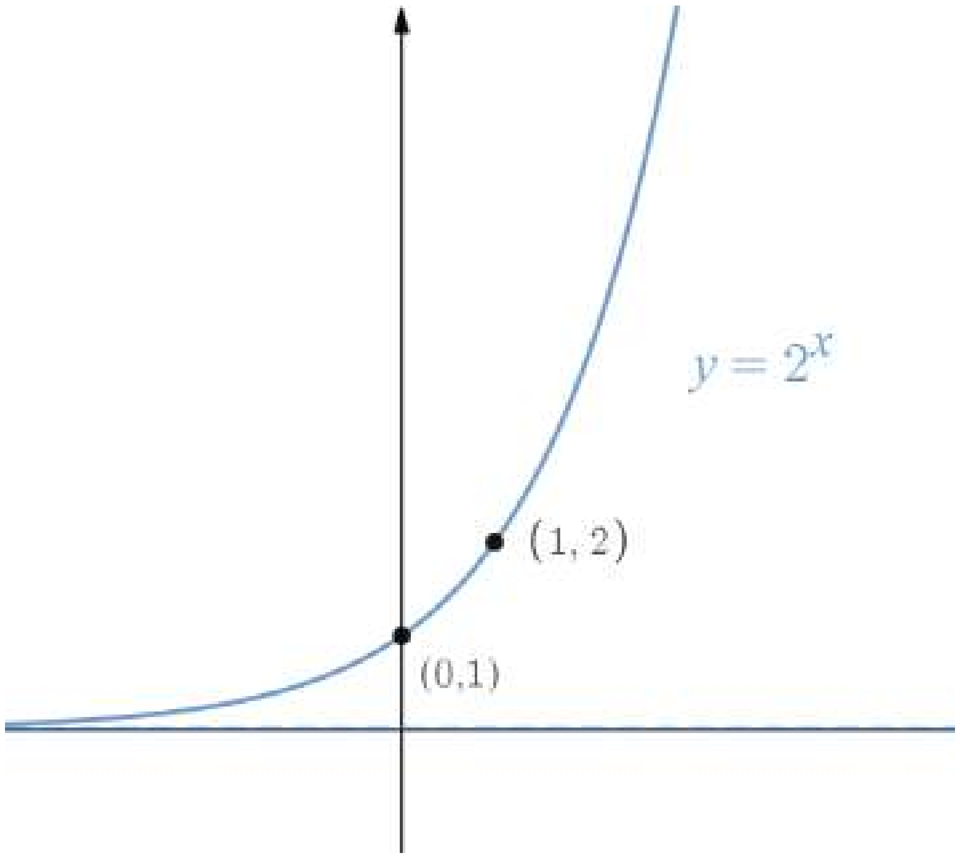
$$\boldsymbol{a}^{\frac{\boldsymbol{m}}{\boldsymbol{n}}} = \sqrt[\boldsymbol{n}]{\boldsymbol{a}^{\boldsymbol{m}}}$$

Exponentials

Graph features:

- Horizontal asymptote at $y = 0$
- y-intercept of 1
- Domain = $\mathbb{R}$, Range = $(0, \infty)$

- Why does this asymptote exist?
- Why does the graph ‘approach’ this asymptote?

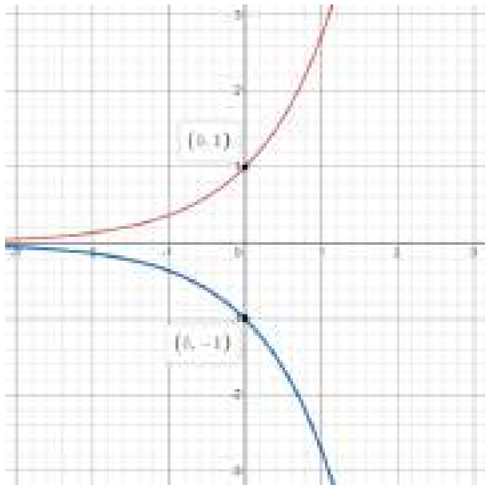


Exponentials

Let's apply our knowledge of transformations!

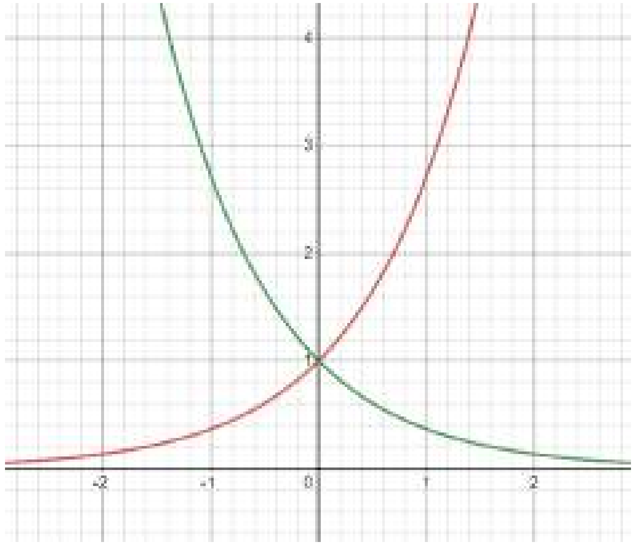
If we reflect the graph in the x-axis:

$$(x, y) \to (x, -y)$$
$$x' = x, \qquad y' = -y$$
$$x = x', \qquad y = -y'$$
$$-y' = e^x \Leftrightarrow y' = -e^x$$



If we reflect the graph in the y-axis:

$$(x, y) \to (-x, y)$$
$$x' = -x, \qquad y' = y$$
$$x = -x', \qquad y = y'$$
$$y' = e^{-x}$$



Exponentials

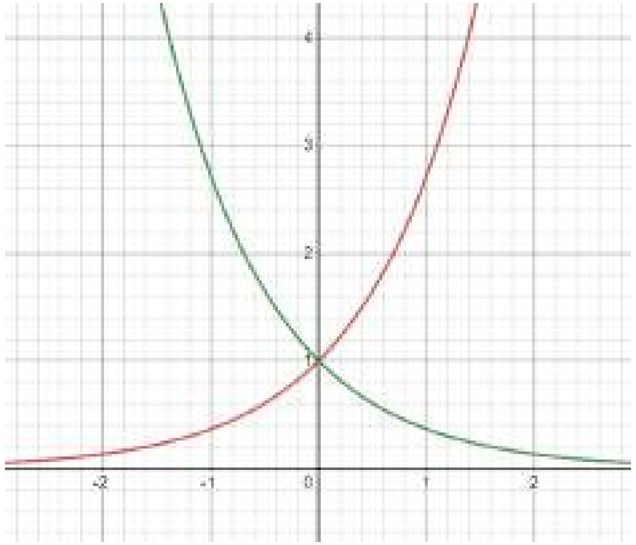
If we reflect the graph in the y-axis:

$$(x, y) \to (-x, y)$$

$$x' = -x, \qquad y' = y$$

$$x = -x', \qquad y = y'$$

$$y' = e^{-x}$$



Applying $a^{mn} = (a^m)^n$:

$$y = e^{-x}$$
$$= (e^{-1})^x$$

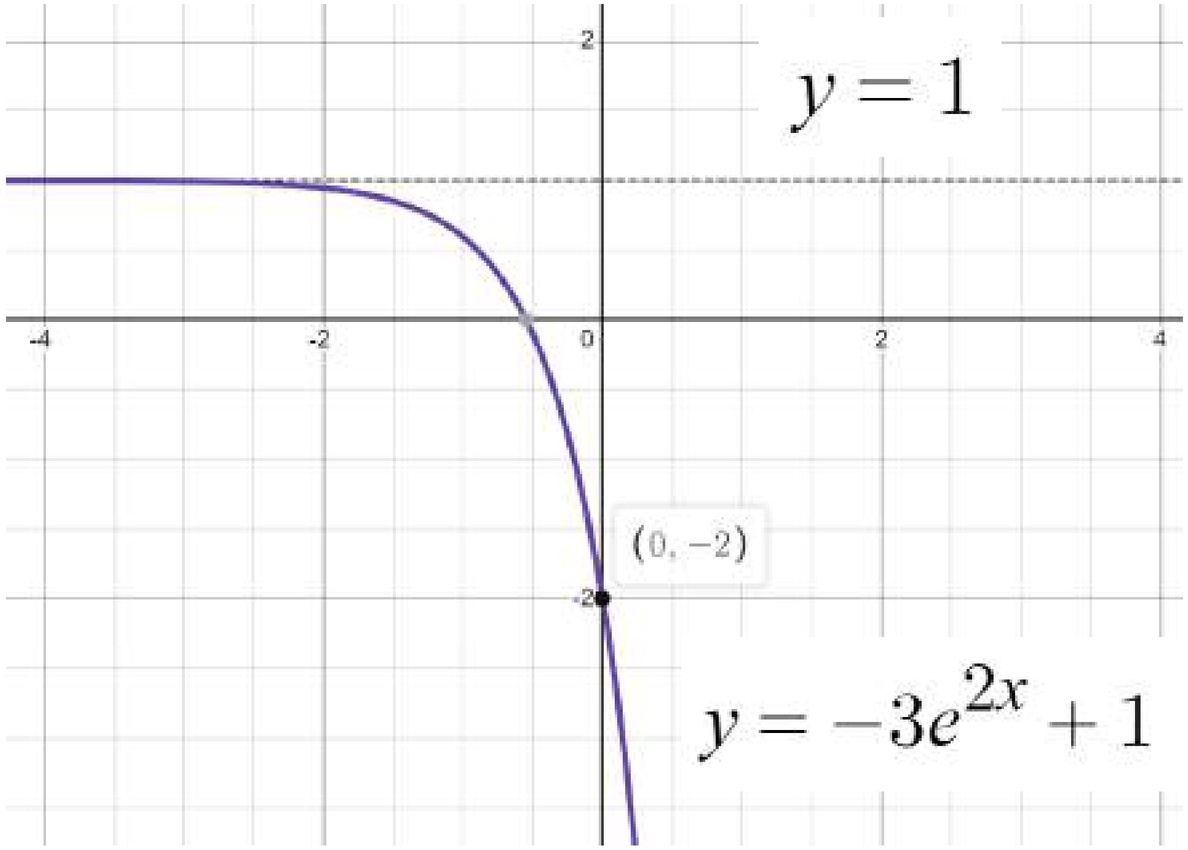
Applying $a^{-m} = \frac{1}{a^m}$:

$$y = (e^{-1})^x$$
$$= \left(\frac{1}{e}\right)^x$$

So whenever you see a fraction, think index laws!

Exponentials

Other transformations are also obviously possible:



Checklist:

- x and y intercepts
- Horizontal asymptote

SOLVING EXPONENTIAL EQUATIONS

Make each side the same base

- Combine everything into one base on each side ($2^2 = 2^x$) then equate the powers; if $a^x = a^y$ then it follows that $x = y$

Put in quadratic form (very common!)

- How can we solve $y = 4^x + 3 \cdot 2^x + 6 = 0$?
- Let $a = 2^x$. Then the equation becomes $a^2 + 3a + 6 = 0$
- Solve for a then sub 2^x back in for a

Use index laws throughout!

Exponentials

Example

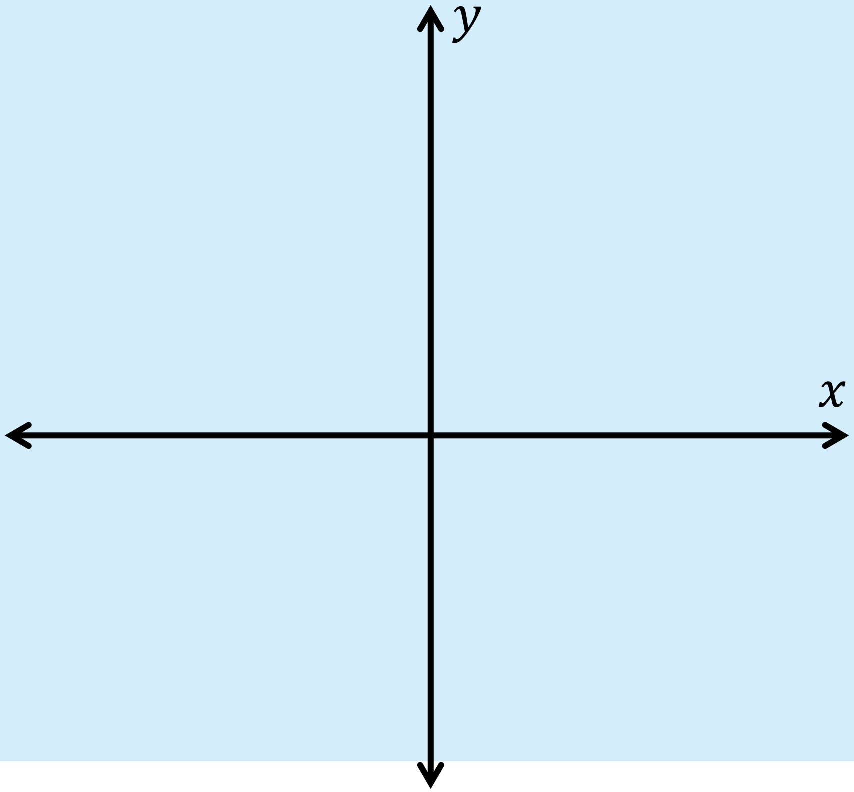
VCAA 2014 Exam 1

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

Example

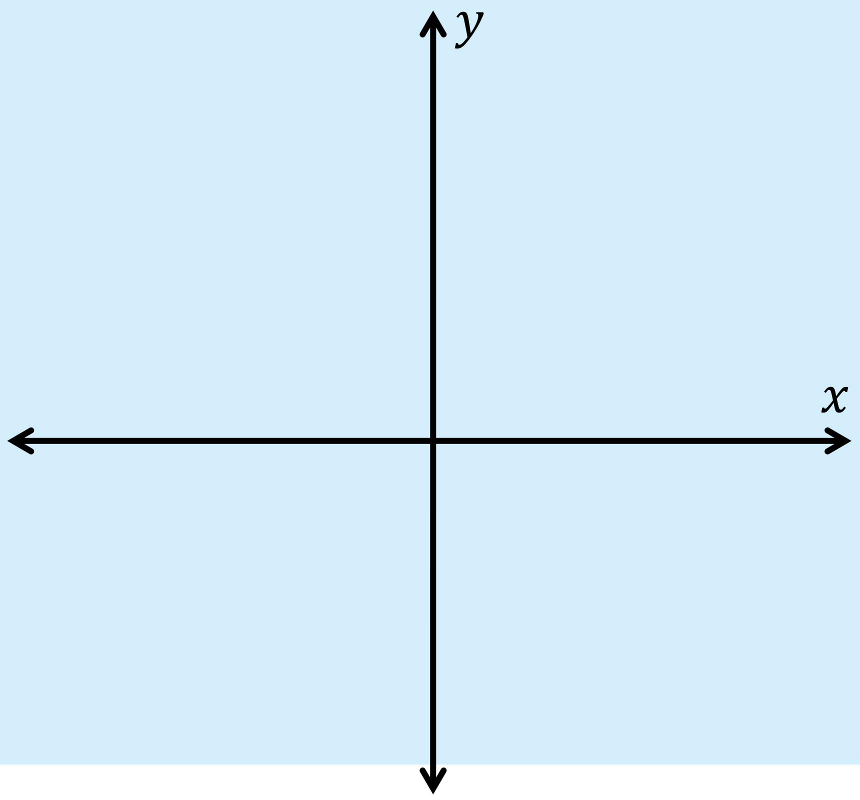
Sketch the following labelling any axis intercepts and asymptotes

a. $y = \left(\frac{1}{4}\right)^{x} - 2$



Sketch the following labelling any axis intercepts and asymptotes

b. $y = e^{2x+3}$



Exponentials

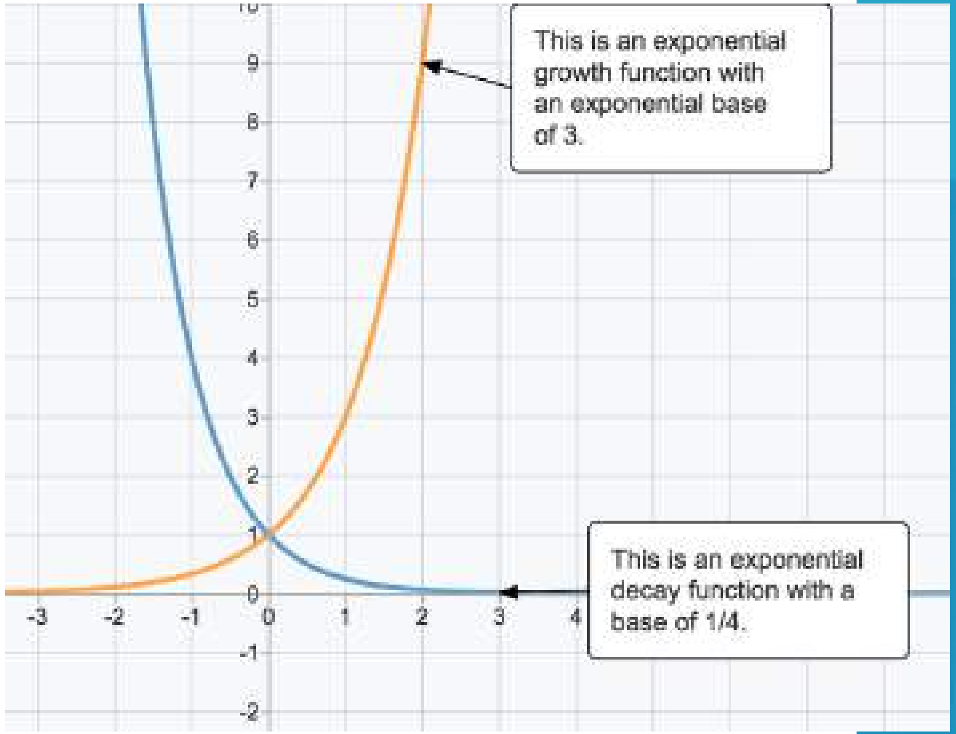
- Exponential functions can model for growth and decay in populations
  - Growth: bacteria numbers, temp of cooling cake

- Often comes in the form: (NEED TO CHECK QN)

$$y = y_0 \times b^{kt}$$

- Where
  - y_0 is the initial value of y (the value when time begins)
  - k is a constant
    - if $k > 0$ the quantity experiences exponential growth
    - if $k < 0$ the quantity experiences exponential decay

If modelling time, domain is $[0, \infty)$.



The temperature of a room was $10°\mathrm{C}$ before Peter put the heater on. After 2 minutes the room was $10 \times \sqrt{2} \approx 14.1°\mathrm{C}$. An equation that models the temperature (T) in the room t minutes after turning on the heater is

$$T = T_0 \times 2^{kt}$$

a. What are the values of T_0 and k?

Logarithms

Defined as the inverse of exponential functions:

Recall that an inverse function essentially undoes a given function!

$$a^x = b \Leftrightarrow \log_a(b) = x$$

- Logarithms with base e are known as the natural log, and can be written as $\log_e(x)$ or $\ln(x)$
  - You can use either notation in the exam
  - Get into the habit of working with $\ln$ as opposed to logs in other bases, e.g. $\log_2$

$$\log_a(m) + \log_a(n) = \log_a(m \cdot n)$$

$$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$$

$$\log_a(m^p) = \ p \cdot \log_a(m)$$

$$\log_a\left(\frac{1}{m}\right) = -\log_a(m)$$

$\log_a 1 = 0$ as $a^0 = 1$

$$\log_a a = 1$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

UNDEFINED:

$\log_a(0)$

$\log_a(-ve)$

Logarithms

$$\log_a b = \frac{\log_c b}{\log_c a} \quad \Leftrightarrow \quad \log_a(b) = \frac{\ln(b)}{\ln(a)}$$

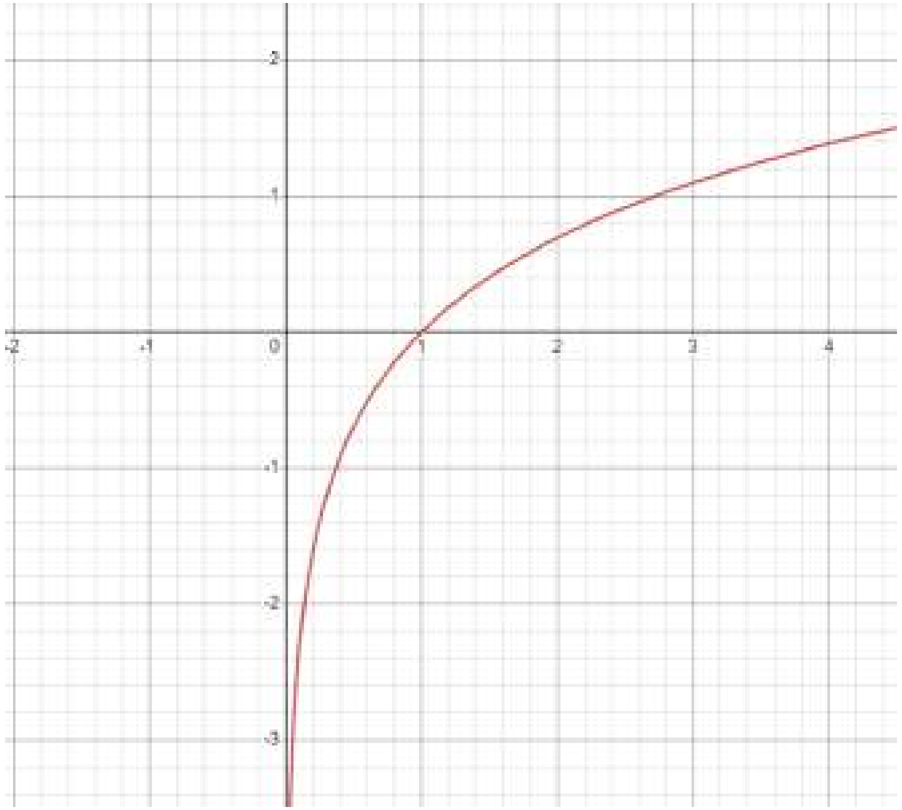
$$\log_a a^x = x$$
$$a^{\log_a x} = x$$
$$\Leftrightarrow \quad \ln(e^x) = e^{\ln(x)} = x$$

Exponentials

Graph features:

- Vertical asymptote at $\mathrm{x} = 0$
- x-intercept of 1
- Domain = $(0, \infty)$, Range = $\mathbb{R}$

- Why does this asymptote exist?
- Why does the graph 'approach' this asymptote?
- How else can we use inverse functions to understand the nature of this graph?

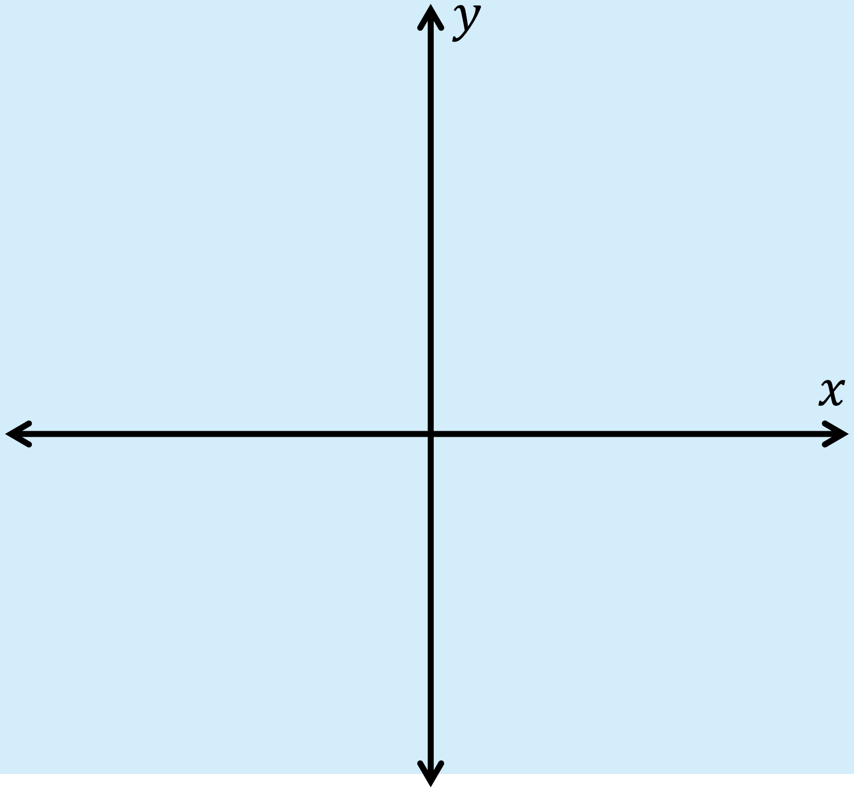


Logarithms

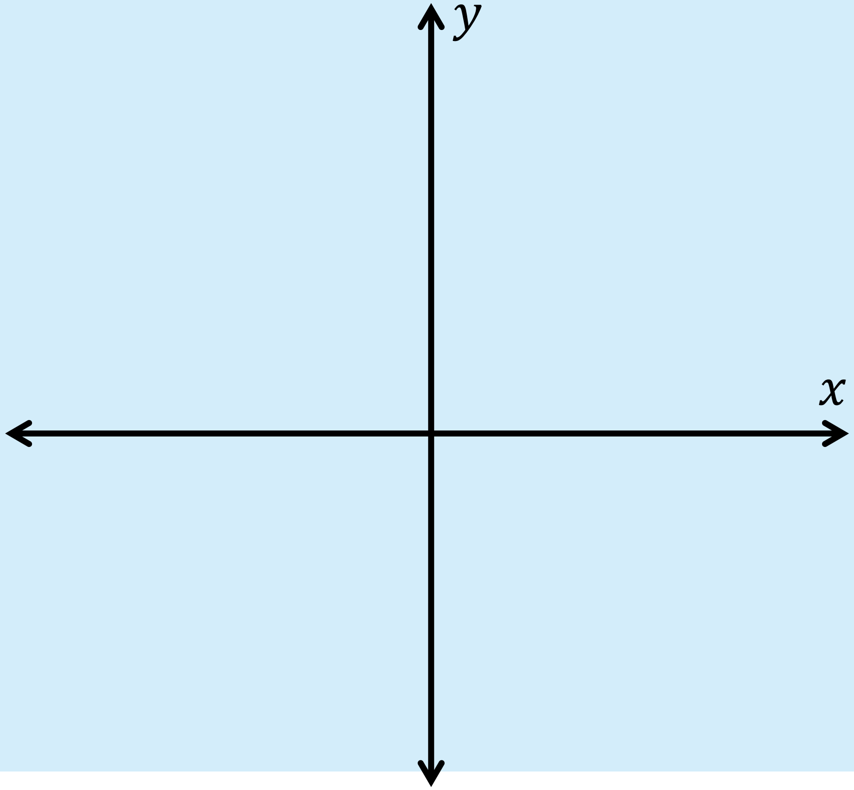
Example

Sketch the following

a. $y = \log_e(x-3)$



b. $y = \log_e\left(\frac{1}{2x}\right) + 1$



Solving log equations

Helps to know these:

- $\log_a(x) = \log_a(y) \Leftrightarrow x = y$
- $e^{\ln(x)} = \ln(e^x) = x$
- $y = \ln(x) \Leftrightarrow e^y = x$
- $\log_a(b) = \dfrac{\ln(b)}{\ln(a)}$

Note that solving exponential equations usually will involve logs, and solving log equations usually will involve exponents. As a rule of thumb:

- When solving exponential equations, usually you will
  - Take the ln of both sides and apply log laws
  - Incite the log at the very end of your working, e.g. $e^x = 4, \therefore x = \ln(4)$
- When solving log equations, usually you will
  - Incite the exponential at the very end of your working, e.g. $\ln(x) = 4, \therefore x = e^4$

VCAA 2015 Exam 1

Solve $\log_2(6-x) - \log_2(4-x) = 2$ for x, where $x < 4$

Logarithms

Example

If $\log_r(p) = q$ and $\log_q(r) = p$, show that $\log_q(p) = pq$.

If $\log_r(p) = q$ and $\log_q(r) = p$, show that $\log_q(p) = pq$.

Bonus: How do I lay out my working?

Don't be afraid to use actual English words to signpost your working out! Some examples are:

- 'by observation, $A(x)$ strictly increases as a increases'
- 'the graphs are now symmetric about the line $y = x + 1$'
- Shorter phrases I more commonly used: 'we require that', 'is given by', '…as required'

The tangent at $(t, p(t))$ has equation $y = (3t^2+w)x - 2t^3$

$\therefore$ it's x-intercept is given by

$$0 = (3t^2+w)x - 2t^3 \iff x = \frac{2t^3}{3t^2+w}$$

We require this to be at $x = t$:

$$t = \frac{2t^3}{3t^2+w} \quad \ldots$$

Often you don't require words to get full marks on the exam, but

- Helps examiner
- Encourages neat working

Bonus: How do I lay out my working?

Be comfortable with mathematical notation used to signpost working and otherwise compact working:

- $\therefore$ = therefore (feel free to use liberally)
- $\Rightarrow$ = implies that (use when invoking a theorem or mathematical fact, e.g. null factor theorem or index law, or when making any other logical conclusion)

$$\begin{aligned} 0 &= x^2 + 4x + 4 \\ &= (x+2)^2 \\ \Rightarrow x &= -2 \end{aligned}$$

- $\Leftrightarrow$ = if and only if (use to skip a whole bunch of intermediate algebra steps which you know won't be worth any marks)

$$\frac{1}{1-p} - 1 > 0 \quad \Leftrightarrow \quad 0 < p < 1$$

- $\in$ = is an element of

Bonus: How do I lay out my working?

Underline important conclusions throughout working, and box your final answer.

- Include units!
- Sometimes the final answer will need to be written in a full sentence so as to answer the question appropriately

$$\frac{35}{60} = \frac{x}{240} + \frac{x}{320}$$

$$\frac{35}{3} = \frac{x}{12} + \frac{x}{16}$$

$$= \frac{7x}{48}$$

$$560 = 7x$$

$$\underline{\underline{x = 80}}$$

total distance $= 2x = 2(80) = 160$

$\therefore$ $\boxed{\text{total distance is 160 km.}}$

If you want it straight:

<u>My super simple advice:</u>

- Have a good foundation
  - Use your teachers, peers, textbooks, internet to understand the basic concepts
  - Don't be afraid to be a nerd! Youtube videos, Wikipedia pages, messing around in Desmos etc.
- **Practice is the only thing that matters** – exposure to questions/time organisation
  - Chapter reviews, past SACs, exam papers, Checkpoints
  - Force your teacher to be on their toes about giving you these
- EXAM PAPERS!!!
  - At least 10 whole papers (exams 1 and 2) – put questions you got wrong + solutions in your bound ref
- Calculator skills save time

Tips for Bound Reference

Pages must be **permanently bound** (not detachable)

- No perforated pages
- Nothing sticking out (sticky notes/loose pages)
- You can combine pages/books if they're bound
- No fold-outs

What to put in it:

- 3 principles for any math subject: Underline, Draw, Check
- Colours! (e.g. blue for titles, red for subtopics, green for examples/calc)
- Have formulas incase you forget but hopefully you will memorise it
- Otherwise most important: TRICKY EXAMPLES/can glue past papers/SACs
- Write final conclusion for each topic: common mistakes, shortcuts, etc

Note: This is only a guideline; find something that works for you!

Conclusion

Today we covered:

Polynomials:

- Quadratics, Cubics, Quartics
- Power, Hyperbola, Truncus
- Sketching, solving, factorizing
- Equating coefficients

Functions:

- Domain, Range
- Hybrid, Composite, Inverse

Transformations:

- How to sketch, describe, solve

Exponentials + Logarithms

- Index and Log laws
- How to solve and sketch

ATARNotes

QUESTIONS?